A TREATISE

ON

OPTICS

BY

S. PARKINSON, D.D.; F.R.S.

FELLOW AND LATE TUTOR OF ST JOHN'S COLLEGE, CAMBRIDGE.

FOURTH EDITION REVISED.

London:

MACMILLAN AND CO.

1884

Cambridge:
PRINTED BY C. J. CLAY, M.A. & SON
AT THE UNIVERSITY PRESS.

PREFACE TO THE FIRST EDITION.

THE present work may be regarded as a new edition of the *Treatise on Optics* by the Rev. W. N. Griffin, which being some time ago out of print, was very kindly and liberally placed at my disposal by the Author. I have freely used the liberty accorded to me, and have rearranged the matter with considerable alterations and additions—especially in those parts which required more copious explanation and illustration to render the work suitable for the present course of reading in the University. The numerous diagrams which the subject requires have been inserted in the body of the work, instead of being collected in plates at the end, and are thus rendered more convenient for reference.

I have appended a collection of examples and problems—distributed under the heads of the several chapters, as well as a miscellaneous set,—which are sufficiently numerous and varied in character to afford a useful exercise for the student: for the greater part of them I have had recourse to the

Examination Papers set in the University and the several Colleges during the last twenty years.

Subjoined to the copious Table of Contents I have ventured to indicate an elementary course of reading, not unsuitable for the requirements of the present examination of the *First Three Days* in the Senate-House.

S. PARKINSON.

ST JOHN'S COLLEGE,
October, 1859.

The FOURTH Edition has been carefully revised and many additions made to it: the collection of Problems in particular has received large accessions from recent Examination Papers.

ST JOHN'S COLLEGE,
May, 1884.

CONTENTS.

CHAPTER I.

PAGE

Laws of Propagation of Light:—Direct Reflexion and Refraction 1

Subject of this treatise, *Geometrical* or *Formal Optics*, Art. 1; Definition of *self-luminous*, *medium*, *rays*, *pencil* of light, incidence *direct* and *oblique*, 2—7; *laws of reflexion* and *refraction*, 8—10; foundation of such laws, 11; Def. of *Geometrical* and *Principal Focus*, *focal length*, *aberration*, 12, 13; *Reflexion* and *refraction* at a *plane surface*, 14—19; Discussion of the *Geometrical focus* and *conjugate foci* of a pencil directly reflected at a spherical surface, 21—25; *Geometrical* and *visible image*, 26; Discussion of a pencil directly refracted at a spherical surface, 27—31; Illustrations,—successive reflexions at *parallel mirrors* and *inclined mirrors*, 32—35; *aplanatic surfaces*, 36.

CHAPTER II.

Illumination of Surfaces . . . 29

Measure and *law* of illumination, 37—40; law of *intensity of emission*, 41; method of calculating illumination, 42, 43; objects appear equally bright at all distances, 44; Ritchie's *photometer*, 45; Foucault's *photometer*, 45*; example, 46; velocity of light, 47; *Foucault's* Experiment, 47*.

CHAPTER III.

Aberration of small direct pencils . . . 42

Measure of aberration, 48; Formula for aberration after *refraction at a plane surface*, 50, 51; after *reflexion at a spherical surface*, 52; after *refraction at a spherical surface*, 53; *Least circle of aberration*, its formation, position and dimensions, 55, 56.

CHAPTER IV.

PAGE

Focal lines of small oblique pencils,—Caustics 53

Astigmatism, caustic curve, 57; mode of determining the form of a caustic, —*catacaustics, diacaustics,* 58; Def. of *centres* of the *surface* and *face*,—incidence *centrical* and *excentrical*—*axis* of a surface, 59; *oblique* pencil reflected at a plane surface, 60; Formation of *focal lines* by an oblique pencil,—Def. of *Primary* and *Secondary focal line*,—*circle of least confusion*, 62—64; Illustration of *geometrical* and *visible image*, 65; Position of the Foci of a small oblique pencil after *reflexion* at a spherical surface, 66; the same after *refraction* at a plane or spherical surface, 67—69; Position and dimensions of the focal lines and circle of least confusion, 70; Illustrations,—caustic after reflexion at a circle in certain cases, 71, 72; caustic after refraction at a plane surface, 73, 74.

CHAPTER V.

Successive reflexions and refractions;—prisms—lenses 73

Successive reflexions at plane surfaces, 77, 78; successive refractions at plane surfaces,—*critical angle*, 79—81; *a plate*, 82; *absolute* and *relative index* of refraction, 83, 84; refraction through a plate, 85—87; Def. of *edge, faces, refracting angle, principal section of a prism*, 88; *deviation* through a prism, *minimum* deviation, 90—95; foci of a small pencil passing through a prism, 96; Def. of a *lens, axis* of a lens, *forms* of lenses, 97, 98; geometrical focus of a pencil passing through a lens, 99, 100; *principal focus, focal length*, of a lens,—*concave* and *convex* lens,—*power* of a lens, 101—104; a sphere regarded as a lens,—its *focal* length, 105, 106; relative motion of conjugate foci in the case of a lens, 107; focal length of a lens determined *practically*, 108; *cylindrical lenses*, 108*; *centre* of a lens, 109; *focal centres* of a lens, 110; Discussion of an oblique pencil passing through a lens, 112, 113; combinations of lenses, 114; approximate determination of the focus of a small oblique pencil, 115; Lenses used in Lighthouses, 117.

CHAPTER VI.

Refraction through media of varying density;—reflexion and refraction at a surface in any manner 107

Absolute and *relative* refractive index, 118; method of determining the path of a ray in a medium of varying refractive power under various

PAGE

circumstances, 120—125; Formulæ for determining the course of a ray reflected or refracted at two or more surfaces in any manner, 126, 127.

CHAPTER VII.

Spherical aberration of lenses;—Excentrical pencils . 119

Aberration of a direct pencil, 129; form of a lens to produce *minimum aberration* for rays parallel,—or not so, 130—132; *examples*, 133; Excentrical pencil passing through one or more lenses—mode of defining it in simple cases, 134—137; *equivalent* lens, 138, 139; examples, Ramsden's and Huyghens' eye-pieces, 140.

CHAPTER VIII.

Images and Caustics 131

Image—*geometrical* or *visible*,—*real* or *virtual*,—*erect* or *inverted*, 141—143; Example, image of a straight line formed by a lens, 144—146; *Distortion*,—*linear*, *angular* and that of *curvature*, 147; Deformation of an image relatively to the object, 148, 149; calculation of the distortion in the case of a spherical mirror, 150; Brightness of an image, 151, 152; length, &c. of a caustic, 153, 154.

CHAPTER IX.

On the Chromatic Dispersion of Light . . 148

Newton's experiment, 156; *pure spectrum*, how observed by Wollaston and Fraunhofer, 157, 158; *fixed lines* of the spectrum, 159—161; *phosphorescence* and *fluorescence*, 161*; *spectrum analysis*, 162 (α), (β), (γ), (δ); mode of determining the *refracting angle* of a prism,—the *minimum deviation* of a given colour—and the *index of refraction*,—*test of the law of refraction*, 163—165; *standard scale* of colours, 166; Def. of *deviation*, *dispersion*, *irrationality of dispersion*, *secondary spectra*, 167; dispersion of two colours by a prism, 168; dispersive power, 171; *achromatic combinations*,—*possible* but *imperfect*, 172, 173; conditions of achromatism for *prisms*, 174—176; for *lenses*, 177, 181; *circle of chromatic aberration*, 182—184; colours of natural bodies, primary colours, 185. Table of refractive indices, 185*.

CHAPTER X.

PAGE

Of Vision and Optical Instruments . 180

The Eye,—retina, crystalline lens, uvea or *iris, punctum cæcum, punctum luteum*, 187—190; *Binocular* vision, 191; *Pseudoscope*, 192; dimensions &c. of the eye, *thaumatrope, anorthoscope*, &c. 194; Defects of vision, *long* and *short* sight, 195; vision through a lens, 197; *visual angle* of an object seen through a lens, 198; *magnifying power* of a lens, 200, 201; Telescopes, their object, 203; the *Astronomical* Telescope, its *field of view*, and *magnifying power,—the ragged edge and stop,—collimator*, 204—206; *Galileo's* Telescope, 207—210; *Newton's* Telescope, 211—213; Illustration of the use of *Newton's* Telescope, 213*; *Herschel's* Telescope, 214—216; *Gregory's* and *Cassegrain's* Telescopes, 217—221; Defects of telescopes, 223—225; *specula* for reflecting telescopes, 226—228; the *finder* of a telescope, 229; object-glasses, 230; Eye-pieces, 231, 232; Astronomical telescope with Huyghens' or Ramsden's Eye-piece, 233—236; the erecting eye-piece, 237; *cross-wires* of an eye-piece, 239; *Dynamometer*, 241—243; *Microscopes*, 244; *simple* and *compound*, 245—248; Wollaston's *microscopic doublet*, 246; magnifying power of a microscope, 249; *the Camera Obscura*, 250; *the Camera Lucida*, 252, 253; *Hadley's Sextant*, 254—256; *the Reflecting Goniometer*, 257; *Brightness* of the image in an Astronomical Telescope, 257*.

CHAPTER XI.

Of the Rainbow, &c. 237

Direct refraction through a sphere of water with *one* or *two* internal reflexions, 259; discussion of the caustic formed by parallel rays incident upon a sphere of water and emergent after one internal reflexion, 260, 261; explanation of the *Primary* Rainbow, 262; the caustic after two internal reflexions within a sphere of water, 264—266; the *Secondary* Rainbow, 267; order of colours and dimensions of the rainbow, 269—272; halos, general remarks, 273; *White* Rainbows, p. 342.

Examples and Problems 255

For an Elementary Course the attention of the Student may be confined to the following articles:—

1—34, 37—41, 44, 45, 57, 58, 65, 74, 76—86, 88—92, 94, 95, 97—111, 114—117, 138 Def., 139—143, 147—149, 186—222. 233—240, 244—252, 254.

OPTICS.

CHAPTER I.

LAWS OF PROPAGATION OF LIGHT:—DIRECT REFLEXION AND REFRACTION.

1. WHEN a material object is presented before us, we become by vision sensible of its existence and figure. In such a case light is said to be propagated from the object to our eyes, and the science of *Optics* has for its design the examination of the circumstances of such propagation.

The science is divided into *Geometrical* and *Physical Optics*. In *Geometrical Optics* the circumstances of the transmission of light are computed on certain laws established by experiment; in *Physical Optics* these laws are accounted for on hypotheses of the structure of bodies, and of the matter filling the space in which they are placed. In a similar manner, in geometrical or plane astronomy the phenomena of heavenly bodies are calculated on observed laws which their apparent motions are found to obey; in physical astronomy these apparent laws are shewn to result from the hypothesis of gravitation.

The former branch of the science is the subject of the present treatise, wherein from certain laws established by experiment under simple circumstances the course of light under more complex circumstances is computed, and the results applied to the construction of optical instruments. These investigations will be conducted in independence of the physical branch of the subject—the experimental laws on which we commence being equally true, whatever be the nature of the hypothesis which professes to account for them.

2. *Def.* A body is said to be *self-luminous* when it is capable in itself of making our eyes sensible of its existence.

Thus the sun, the stars, a lamp, a red-hot iron, &c., are self-luminous, but by far the greater number of natural bodies possess no such property. Such bodies are luminous only by reflexion and require the presence of another luminous body to render them visible. Thus when a lamp is brought into a dark room, the other bodies in the room become visible, and become more or less capable of illuminating others; but this property ceases and they become invisible when the lamp is extinguished or withdrawn.

Obs. When *a luminous point* or *an origin of light* is mentioned, we may understand it to be a minute portion of a luminous surface, in no direction perceptibly extended: and so by *a luminous line* we may understand such a surface perceptibly extended in one direction only. We shall regard them simply as a *mathematical point* and a *mathematical line* respectively.

3. *Def.* Any space or substance which light can traverse is called a *medium*. Such as *a vacuum*, *glass*, *water*, &c.

In this Treatise we shall consider only *media* which are not *crystallized*.

4. *Def.* When light emanates from a luminous point, we regard it as made up of *rays*, understanding by a *ray* the smallest portion of light which can be separately transmitted, stopped, or reflected; and we shall treat such rays as mathematical lines.

In a uniform medium, it will be assumed that the course of a ray of light is a straight line; this law is shewn by experiment to be true in general. There are certain cases of apparent exception, such as are presented by the phenomena of diffraction, the explanation of which belongs to physical optics.

5. *Def.* An assemblage of rays proceeding from a luminous point, is a *pencil* of light.

The *form* of a pencil, unless an exception be expressly mentioned, will be regarded as a *right cone* having the origin of light for the vertex, and when this origin is infinitely distant, this cone becomes a circular cylinder as its limiting form. The geometrical axis of the cone or cylinder is called the *axis* of the pencil.

If the rays of a pencil of light produced in a direction opposite to that of propagation meet in a point, the pencil is *divergent;* if the origin be infinitely distant, its limiting form is a pencil of *parallel* rays; if the rays produced in direction of propagation meet in a point, the pencil is *convergent.*

The degree of divergence or convergence may be measured by the vertical angle of the cone which the pencil forms.

6. *Def.* When a pencil meets the surface of any substance or medium, the incidence is called *direct,* if the axis of the pencil coincides with the normal to the surface at the point of incidence: in other cases the incidence is called *oblique.*

7. When light is incident on the surface of a medium different from that in which it is proceeding, a portion of it is *scattered* or *dispersed* over the surface and makes the surface visible; another portion is in general *reflected* in the first medium (according to a law which will presently be stated)—and in certain cases (as when the medium upon which it is incident is translucent) a third portion enters this new medium according to another law, and is said to be *refracted.*

The course of the reflected and refracted rays where both exist, may be considered separately.

8. *Law of Reflexion.*

When a ray is reflected at the surface of a medium,

(i) The incident and reflected rays lie in one and the same plane with the normal to the surface at the point of incidence, and on opposite sides of the normal.

(ii) The angles which the incident and reflected rays make with the normal to the surface at the point of incidence are equal.

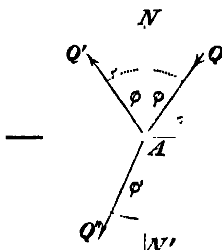
9. *Law of Refraction.*

When a ray is refracted at the surface of a medium,

(i) The incident and refracted rays lie in one and the same plane with the normal to the surface at the point of incidence and on opposite sides of it.

(ii) The sines of the angles which the incident and refracted rays make with the normal to the surface at the point of incidence, have a ratio depending only on the media between which the refraction takes place, and the nature of the light.

10. Thus if a ray QA be incident at A on the surface of a medium (reflecting or refracting or both): NAN' the normal to the surface at A; AQ', AQ'' the reflected and refracted rays respectively; then will the lines QA, $Q'A$, $Q''A$, NAN' all lie in one plane, and the $\angle\, QAN = \angle\, Q'AN$. Also, if we write $\angle\, QAN = \phi$, $\angle\, Q''AN' = \phi'$ then will the ratio $\frac{\sin\phi}{\sin\phi'}$ be constant at whatever angle of incidence a ray of the same kind of light is refracted from one given medium into another.



This constant quantity $\frac{\sin\phi}{\sin\phi'}$, is usually denoted by the symbol μ, and is called the refractive index between the two media for the particular species of light considered. It is a *parameter* which varies, (i) if the nature of the light be altered, (ii) if the relation between the two media be altered.

The relation between ϕ and ϕ' is commonly written

$$\sin\phi = \mu \sin\phi'.$$

It will *in general* be supposed in these pages that the refraction takes place into a denser medium, in which case μ is greater than unity, and the angle of refraction less than the angle of incidence {i. e. $\phi' < \phi$}.

11. *Remark.*

The process by which these and physical laws in general are established experimentally is this. Direct experiments render the law probable; such experiments however are seldom made with such minute accuracy as to prove the law exactly true;—next on supposition of the truth of the law in question,

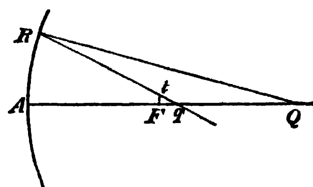
the circumstances of more complex phenomena are computed, and when the results of these computations are found in repeated instances of various kinds to agree minutely with observations, we have a very high degree of probability of the strict truth of the law. It is important to observe that of the truth of the laws of physical science we have only a moral certainty,—a certainty arising only from the improbability that an untrue principle should happen to explain successfully a great variety of phenomena. The first law of motion in Dynamics, for example, we think probably true from experiments made on bodies on the Earth; it is proved by the agreement of the motion of the heavenly bodies calculated on supposition of its truth, with the motions which they are observed to have. On such foundations all the laws of natural philosophy rest.

12. Various illustrations of the law of reflexion have been given, more or less accurate; perhaps one of the highest confirmations of it is derived from the accordance of *Transit observations* of Stars made by reflexion at the surface of mercury with results obtained independently of reflexion. Of the law of refraction we shall speak hereafter when we come to explain the mode of determining the refractive indices of different substances.

The law of reflexion seems to have been known in very early times as we find it laid down in the earliest writers on Optics; the law of refraction was first accurately ascertained by Snell, about A.D. 1621, though it was first published by Descartes.

13. *Direct Reflexion and Refraction.*

Def. Let Q be the origin of a pencil of rays whose axis QA is incident directly (Art. 6) at A on a plane or spherical reflecting or refracting surface.



Then QA is the axis of the reflected or refracted pencil. Let QR be any ray incident at R whose direction after re-

flexion or refraction (produced if necessary) cuts the axis in q, since the normal at R lies in the plane QAR. Then as R is taken nearer and nearer to A, the point q will approach some point F in the axis, as its limiting position; and by taking R sufficiently near to A the distance Fq may be made less than any assignable magnitude. This limiting position (F) of the point q is called the *Geometrical Focus* of the reflected or refracted pencil.

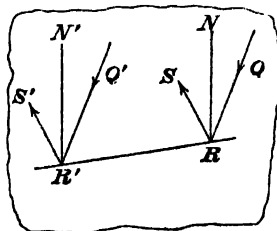
The point F will in some cases be nearer the surface than q is; in others more remote.

Def. The *principal focus* of a spherical reflecting or refracting surface is the geometrical focus of a pencil of parallel rays incident directly upon the surface parallel to a fixed diameter, called the *axis* of the surface.

Def. The *focal length* of a spherical reflecting or refracting surface is the distance between the surface and the principal focus,—measured along the axis.

The distance Fq is called the *aberration* of the ray Rq. If Ft be drawn perpendicular to the axis QA to meet Rq in t, it is sometimes convenient to distinguish Fq as the *longitudinal aberration* and Ft as the *lateral aberration.*

14. *A pencil of parallel rays will consist of parallel rays after reflexion at a plane surface.*



Let QR, $Q'R'$ be any two rays of a pencil of parallel rays incident on a plane reflecting surface at the points R, R'. Let RS be the direction of the ray QR after reflexion. Draw RN, $R'N'$ perpendicular to the surface at the points R, R', and (i) if the planes QRN, $Q'R'N'$ are *coincident*, draw $R'S'$ in this plane parallel to RS;

$$\text{then since } \left.\begin{matrix} R'N' \\ R'S' \end{matrix}\right\} \text{ are parallel to } \left\{\begin{matrix} RN, \\ RS \end{matrix}\right. ;$$

$$\therefore \angle N'R'S' = \angle NRS = \therefore \angle QRN = \angle Q'R'N',$$

and therefore $R'S'$ is the direction of $Q'R'$ after reflexion; but (ii) if the planes QRN, $Q'R'N'$ do not coincide, let $R'S'$ be the intersection of the plane SRR' with the plane $Q'R'N'$.

Now $\left.\begin{matrix} QR \\ RN \end{matrix}\right\}$ are parallel to $\left\{\begin{matrix} Q'R' \\ R'N' \end{matrix}\right.$;

$\therefore$ the plane QRS is parallel to the plane $Q'R'S'$, (Euc. XI. 15),
.......st. line RS .... st. line $R'S'$, (Euc. XI. 16).

Also because $\left.\begin{matrix} QR \\ RN \end{matrix}\right\}$ are parallel to $\left\{\begin{matrix} Q'R' \\ R'N' \end{matrix}\right.$;

$$\therefore \angle QRN = \angle Q'R'N', \text{ (Euc. XI. 10).}$$

Similarly, $\angle SRN = \angle S'R'N'$.

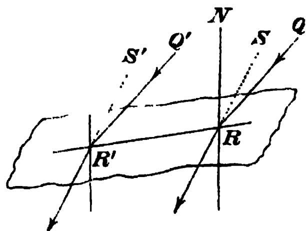
But, $\angle QRN = \angle SRN$, (Art. 8),

$$\therefore \angle Q'R'N' = \angle S'R'N',$$

and $R'S'$ lies in the plane $Q'R'N'$; therefore it is the direction of the ray $Q'R'$ after reflexion; and it has been proved to be parallel to RS. But QR, $Q'R'$ are by supposition *any* two rays of the incident pencil; therefore the reflected pencil consists of parallel rays.

15. *A pencil of parallel rays consists of parallel rays after refraction at a plane surface.*

Let QR, $Q'R'$ be any two rays of a pencil of parallel rays incident on a plane refracting surface at the points R, R'. Let SR be the direction of the ray QR after refraction. Draw RN, $R'N'$ perpendicular to the surface at the points R, R', and let $S'R'$ be the intersection of the plane SRR' with the plane $Q'R'N'$ (if these planes be not coincident).



Now $\left.\begin{matrix} QR \\ RN \end{matrix}\right\}$ are parallel to $\left\{\begin{matrix} Q'R' \\ R'N' \end{matrix}\right.$;

$\therefore$ the plane QRS is parallel to the plane $Q'R'S'$;

$\therefore$ the straight line SR is parallel to the straight line $S'R'$.

Also because $\left.\begin{matrix}QR\\RN\end{matrix}\right\}$ are parallel to $\left\{\begin{matrix}Q'R'\\R'N'\end{matrix}\right.$;

$$\therefore \angle\, QRN = \angle\, Q'R'N'.$$

Similarly, $\qquad \angle\, SRN = \angle\, S'R'N'$;

$$\begin{aligned}\therefore \sin \angle\, Q'R'N' &= \sin QRN \\ &= \mu \sin SRN \\ &= \mu \sin S'R'N'.\end{aligned}$$

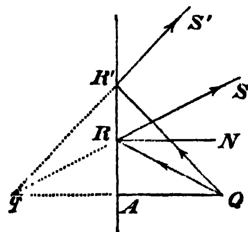
And $S'R'$ lies in the plane $Q'R'N'$, therefore $S'R'$ is the direction of the ray $Q'R'$ after refraction, and it has been proved parallel to SR.

If the planes SRR', $Q'R'N'$ be coincident, and $S'R'$ be drawn in this plane parallel to SR, it may readily be shewn that $S'R'$ is the direction of $Q'R'$ after refraction.

Now QR, $Q'R'$ are *any* two rays of the incident pencil, therefore the refracted pencil consists of parallel rays.

Remark. In the above figure the *dotted* lines SR, $S'R'$ are the directions of the refracted rays *produced backward*: it may not be amiss to suggest to the student that he will often avoid confusion in optical diagrams if he indicates by *dotted lines* any part of the directions of rays which are not *actually* traversed by the rays.

16. *A pencil is incident directly upon a plane reflecting surface, to find its form after reflexion.*



Let Q be an origin of light from which a pencil whose axis is QA is incident directly at A on a plane reflecting surface, QR any ray of the pencil incident at R, and reflected in direction RS in a plane passing through QR, and RN the normal to the surface at R. Since by the law of reflexion SR, QA lie in one plane, let them be produced to meet in q. Then since RN, qQ are parallel

$$\angle RQA = \angle QRN$$
$$= \angle SRN \text{ (Art. 8)}$$
$$= \angle RqA;$$

therefore the angles $\left.\begin{matrix} RqA \\ RAq \end{matrix}\right\}$ are equal to $\left\{\begin{matrix} RQA \\ RAQ \end{matrix}\right.$ each to each, and RA is common to the two triangles;

$$\therefore Aq = AQ.$$

Now QR is *any* ray of the incident pencil; therefore the directions of all the rays of the reflected pencil (produced backward) pass through the point q.

Hence, the form of the reflected pencil is that of a cone whose axis is AQ, and vertex the point q—equidistant with Q from the reflecting surface, and on the opposite side of it.

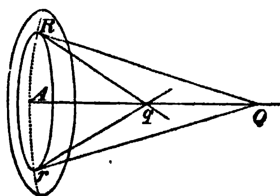
COR. 1. Since the angles RqA, RQA are equal, the *divergence* of the incident and reflected pencils is the same.

COR. 2. If the incident pencil be *convergent*, a similar investigation will shew, that it will *converge* after reflexion to a point equidistant from the surface with the point of convergence of the incident pencil, and on the opposite side of it,—the degree of convergence being unaltered.

COR. 3. The direction of the ray QR after reflexion cuts the axis in q, and the same is true in the limit when R moves up to A; therefore q is the geometrical focus of the reflected pencil.

17. The succeeding cases of direct reflexion and refraction have greater difficulty than the last investigation because in none of them will the pencil after reflexion or refraction pass accurately through a point. Our attention will at present be confined to plane and spherical surfaces.

A pencil whose origin is Q and axis QA incident directly on a plane or spherical reflecting or refracting surface may be regarded as composed of a series of conical surfaces or shells of rays, as QRr, with a common vertex Q and common axis QA. The rays of this conical surface will all be reflected or refracted similarly with respect



to QA, and therefore their directions after reflexion or refraction will form another conical surface with vertex q and axis Aq. Thus the reflected or refracted pencil will consist of a series of conical surfaces with a common axis, but different vertices. The limiting position of the vertex q is the geometrical focus of the pencil. (Art. 13.)

18. The determination of the form of a direct pencil after reflexion or refraction will consist of two parts:

(i) The determination of the geometrical focus.

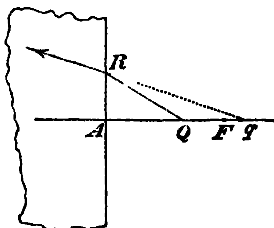
(ii) The determination of the vertex q of any one of the cones of rays above described.

We will confine ourselves in the present chapter to the first of these parts—reserving the second for a subsequent chapter.

Obs. As it is obvious that a ray of light would traverse the same path *backward* if its course were *reversed* in any medium, it will be easily seen that Q (fig. Art. 13) would be the *geometrical focus* of a pencil of rays emanating from F the geometrical focus of Q. In this point of view, *the origin of a pencil* and *its geometrical focus* after reflexion or refraction are *convertible*, and we shall, for brevity, sometimes speak of them as *conjugate foci*.

19. *To find the geometrical focus of a pencil after direct refraction at a plane surface.*

Let Q be the origin of a pencil whose axis QA is incident directly at A on a plane refracting surface, then QA is the axis of the refracted pencil.



Let QR be any ray incident at R and refracted in a direction which, when produced backward, cuts the axis in q. Let F be the geometrical focus, or limiting position of q.

Let $AQ = u$, $AF = v$, lines being considered positive when measured from A in a direction contrary to that of the incident light.

Now RQA, RqA being equal to the angles of incidence and refraction of the ray QR,

$$\sin RQA = \mu . \sin RqA;$$

$$\therefore \frac{AR}{RQ} = \mu . \frac{AR}{Rq};$$

$$\therefore Rq = \mu . RQ.$$

In the limit when R moves up to A, and q to F,

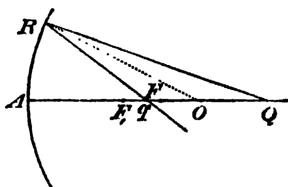
$$AF = \mu . AQ, \text{ or } v = \mu u,$$

which formula gives the position of the geometrical focus of the refracted pencil.

20. *Obs.* In this proposition the pencil is considered *divergent*, as will be done in other cases, but attention to the algebraic sign of u will make the result applicable to a *convergent* pencil. In such a case u is negative and $v = \mu u =$ a *negative quantity*, indicates that the geometrical focus lies in a negative direction, i.e. behind the surface, at a distance from it determined by the numerical value of μu. The student will have little difficulty in drawing a suitable diagram and investigating the case independently.

21. *To find the geometrical focus of a pencil of rays after direct reflexion at a spherical surface.*

Let Q be the origin of a pencil of light whose axis QA is incident directly on a spherical reflecting surface of which O is the centre: then AQ is the axis of the reflected pencil. (Art. 17.)



Let QR be any ray incident at R, and reflected in direction Rq, cutting the axis in q; F the geometrical focus. Join OR.

Let $AQ = u$, $AO = r$, $AF = v$, lines being considered positive when measured from A in a direction contrary to that of the incident pencil.

Now $\angle QRO = \angle qRO$; $\therefore \frac{QR}{Rq} = \frac{QO}{Oq}$, (Euc. VI. 3),

or ultimately when R moves up to A, and q to F,

$$AQ.OF = AF.OQ,$$

or
$$u(r-v) = v(u-r);$$

$$\therefore \frac{1}{v} - \frac{1}{r} = \frac{1}{r} - \frac{1}{u};$$

$$\therefore \frac{1}{v} + \frac{1}{u} = \frac{2}{r},$$

a formula which determines the position of F.

COR. 1. If u be indefinitely great, *or* the incident pencil consist of parallel rays, the formula becomes

$$\frac{1}{v} = \frac{2}{r}, \text{ or } v = \frac{r}{2},$$

which assigns the position of the *principal focus* of the reflector, viz. at a point on its axis equidistant from the *centre* and the *surface*.

If F_1 be the principal focus, $AF_1 = F_1O = \frac{r}{2}$.

COR. 2. Since $\frac{1}{v} + \frac{1}{u} = \frac{2}{r}$, we get by reduction

$$uv - (u+v)\frac{r}{2} = 0; \text{ and } \therefore \left(u - \frac{r}{2}\right)\left(v - \frac{r}{2}\right) = \left(\frac{r}{2}\right)^2.$$

If F_1 be the *principal focus* this result may be written

$$QF_1.FF_1 = AF_1^2,$$

or
$$FF_1 : AF_1 :: AF_1 : QF_1,$$

a geometrical form of the relation which exists between the positions of the *conjugate foci* Q, F.

22. The geometrical result of Art. 21, Cor. 2, can easily be obtained independently; thus, suppose a ray $Q'R$ incident *parallel to the axis* QA to be reflected in direction Rq' (these lines $Q'R$, Rq' are not drawn in the figure), then in the two triangles $Qq'R$, $Rq'q$, we have

$$\angle q'Rq = \angle QRQ' = \angle RQq',$$

and $\angle Rq'q$ is common to the two triangles, which are therefore similar, and we have

$$qq' : q'R :: q'R : q'Q;$$

and therefore ultimately when R moves up to A, and q, q' to F, F_1,

$$FF_1 : AF_1 :: AF_1 : QF_1.$$

Obs. Since u, v are symmetrically involved in the equation $\frac{1}{v} + \frac{1}{u} = \frac{2}{r}$, if F were the origin of a pencil, Q would be its geometrical focus after reflexion at the spherical mirror. The points Q and F are thus convertible, and for this reason are sometimes called *conjugate foci;* as was remarked before, Art. 18, *Obs.*

23. The case of a divergent pencil and concave mirror by which the propositions of Art. 21 have been investigated, is chosen because in it all the lines are measured in a positive direction. It will be seen that attention to the *signs* of u and r will make this case include every other. We suppose in these pages light to proceed from *right* to *left*, and *positive* lines consequently to be measured from *left* to *right;* hence

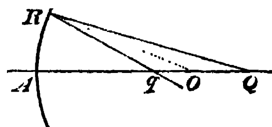
(i) If the surface be *convex*, or O to the left of A, r is negative.

(ii) If the pencil be *convergent*, or Q to the left of A, u is negative.

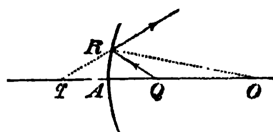
The *sign* which v has in the result in any case, determines on which side of the point A the geometrical focus of the pencil lies, and the *magnitude* of v gives its distance from A.

The four cases of the proposition are here subjoined as exercises for the student to investigate independently.

(i) Divergent pencil,
Concave mirror,
$\therefore \begin{cases} u \text{ is positive,} \\ r \text{ is positive.} \end{cases}$

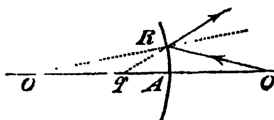


F lies to the *right* or *left* of A according as $u \gtrless \frac{r}{2}$.



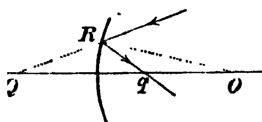
(ii) Divergent pencil, Convex mirror, $\left.\begin{matrix}\\ \end{matrix}\right\}$ $\left\{\begin{matrix} u \text{ is positive,} \\ r \text{ is negative.}\end{matrix}\right.$

F always lies to the left of A.



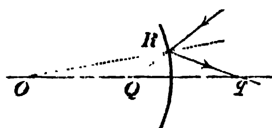
(iii) Convergent pencil, Concave mirror, $\left.\begin{matrix}\\ \end{matrix}\right\}$ $\left\{\begin{matrix} u \text{ is negative,} \\ r \text{ is positive.}\end{matrix}\right.$

F always lies to the right of A.

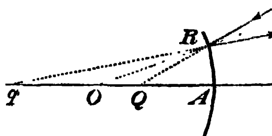


(iv) Convergent pencil, Convex mirror, $\left.\begin{matrix}\\ \end{matrix}\right\}$ $\therefore \left\{\begin{matrix} u \text{ is negative,} \\ r \text{ is negative.}\end{matrix}\right.$

F lies to the right or left of A as u is $< r$

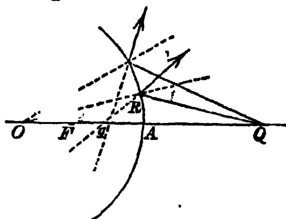


Obs. All these results will follow from a discussion of the equation $\frac{1}{v} + \frac{1}{u} = \frac{2}{\ }$



24. It is sometimes convenient to determine the position of the geometrical focus of a reflected pencil by its *distance from the centre* of the reflecting surface, instead of its distance from the point of incidence.

Let Q be the origin of a pencil of light whose axis QA is incident directly at A on a spherical reflecting surface whose centre is O. Then AQ is the axis of the reflected pencil.



Let QR be any ray incident at R and reflected in a direction which cuts OQ in q, F the geometrical focus of the reflected pencil.

Let $OQ = p$, $OA = r$, $OF = q$, lines being considered positive when measured from O in a direction opposite to that of the incident pencil.

Also let $\angle ROA = \theta$, $\phi = \angle$ of incidence or reflexion of QR.

Then $$\frac{r}{Oq} = \frac{\sin RqA}{\sin ORq} = \frac{\sin(\phi+\theta)}{\sin\phi},$$

$$\frac{r}{p} = \frac{\sin RQA}{\sin ORQ} = \frac{\sin(\phi-\theta)}{\sin\phi};$$

$$\therefore \frac{r}{Oq} + \frac{r}{p} = \frac{\sin(\phi+\theta)+\sin(\phi-\theta)}{\sin\phi} = 2\cos\theta.$$

In the limit $Oq = q$, and $\cos\theta = 1$;

$$\therefore \frac{1}{q} + \frac{1}{p} = \frac{2}{r};$$

a formula which gives the position of F.

Note. This result may easily be deduced from the fact that OR bisects the exterior $\angle QRq$—from which we get

$$OQ : Oq = QR : Rq;$$

ultimately $$OQ : OF = QA : AF,$$

or $$p : q = p - r : r - q,$$

whence $$pr - pq = pq - qr;$$

$$\therefore pr + qr = 2pq;$$

$$\therefore \frac{1}{q} + \frac{1}{p} = \frac{2}{r}.$$

Obs. It will be observed that in investigating the above equation the case of a convex mirror has been taken in order that all the lines which enter may be measured in a positive direction. The remarks of Art. 23 will, *mutatis mutandis,* apply to this.

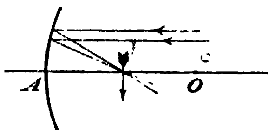
The result $\frac{1}{q}+\frac{1}{p}=\frac{2}{r}$ can of course be deduced from the result of Art. 21, viz. $\frac{1}{v}+\frac{1}{u}=\frac{2}{r}$.

25. *To trace the relative change of position of the conjugate foci of a reflected pencil.*

Suppose the reflecting surface concave, then the formula connecting the positions of the conjugate foci is $\frac{1}{v}+\frac{1}{u}=\frac{2}{r}$, where the lines u, v, r are taken positive *to the right.*

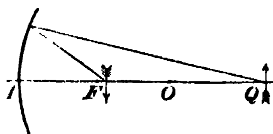
It is clear from the formula that if u increases, v must decrease and *vice versa,* i.e. Q, F *always move in opposite directions.*

(i) Suppose $u=\infty$, i.e. the incident rays parallel, then $v=\frac{r}{2}$ and F coincides with the middle point of AO, the principal focus.

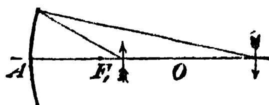


(ii) As Q moves from an indefinite distance up to O, F also moves up to O, and Q, F would coincide at O.

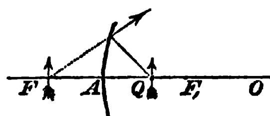
$\therefore$ if $u=r$ we must also have $v=r$.



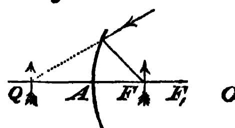
(iii) As Q moves from O to F_1, the middle point of AO, F moves off from O towards ∞, and v becomes ∞ when $u=\frac{r}{2}$, or Q coincides with F_1.



(iv) As Q continues to move in the same direction, from F_1 to A, v becomes negative and approaches from $-\infty$ towards A; and Q, F coincide at A.



(v) When u becomes negative and Q moves from A to $-\infty$, v becomes positive again and F moves from A towards F_1, and F coincides with F_1 when u becomes $-\infty$.



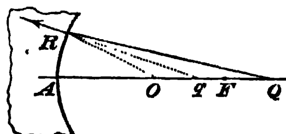
Similarly the change of relative position may be traced when the reflecting surface is *convex*.

26. *Obs.* In the diagrams of the preceding Art. a small object has been placed in the position of Q, and its *geometrical image* in the position of F, as a guide to the student in tracing the form of such an image relatively to the object: understanding by *geometrical image*, the locus of the geometrical foci of pencils diverging from consecutive points of the object, and reflected at the surface.

This *geometrical image*, will rarely coincide with the *visible image* as seen by an eye in a given position. In a subsequent chapter we propose to give a method of determining the visible image in ordinary cases.

27. *To find the geometrical focus of a pencil of rays after direct refraction at a spherical surface.*

Let Q be the origin of a pencil of light whose axis QA is incident directly on a spherical refracting surface of which O is the centre: then QA is the axis of the refracted pencil (Art. 17). Let QR be any ray incident at R and refracted in a direction which cuts QA in q; F the geometrical focus. Join OR. Let $AQ=u$, $AO=r$, $AF=v$, lines being considered positive when measured from A in a direction contrary to that of the incident pencil.



$$\text{Now } \mu = \frac{\sin QRO}{\sin qRO} = \frac{\sin QRO \,.\, \sin ROq}{\sin ROQ \,.\, \sin qRO} = \frac{QO}{RQ} \cdot \frac{Rq}{Oq}.$$

$$\text{In the limit } \mu = \frac{QO \,.\, AF}{AQ \,.\, OF}$$

$$\text{or} \quad \mu u (v - r) = (u - r) v;$$

$$\therefore \mu \left(\frac{1}{r} - \frac{1}{v}\right) = \frac{1}{r} - \frac{1}{u},$$

$$\text{or} \quad \frac{\mu}{v} - \frac{1}{u} = \frac{\mu - 1}{r},$$

a formula which determines the position of F.

Cor. 1. The position of the *principal focus* is obtained by making $u = \infty$, and we get

$$v = \frac{\mu}{\mu - 1} \,.\, r.$$

Cor. 2. To compare the degree of divergence of the pencil *after* and *before* refraction we must take the limit of the ratio $\angle RqA : \angle RQA$ when R approaches A;

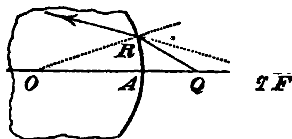
$$\text{this ratio} = \frac{\sin RqA}{\sin RQA} \text{ ultimately} = \frac{RQ}{Rq} \text{ ultimately} = \frac{AQ}{Aq} = \frac{u}{v}.$$

If this result is negative it shews that the incident and refracted pencils are one *divergent* and the other *convergent.*

28. *Obs.* The remarks in Art. 23 in the corresponding problem of reflexion will, *mutatis mutandis,* apply in the case of refraction of Art. 27; and the student is recommended as an exercise to draw the diagrams and go through the investigation for each case that can occur,—with the surface *concave* or *convex* and the incident pencil *divergent* or *convergent.*

29. The following mode of expressing the position of the geometrical focus with respect to the *centre* of the refracting surface is sometimes convenient.

To find the distance of the geometrical focus of a pencil of rays from the centre after direct refraction at a spherical surface.



Let Q be the origin of a pencil of light whose axis QA is incident directly at A on a spherical refracting surface whose centre is O: then QA is the axis of the refracted pencil. Let QR be any ray, incident at R and refracted in a direction which cuts AQ in q; F the geometrical focus.

Let $OQ = p$, $OA = r$, $OF = q$, lines being considered positive when measured from O in a direction contrary to that of the incident pencil.

$$\text{Then } \mu = \frac{\sin QRO}{\sin qRO} = \frac{\sin QRO}{\sin ROQ} \cdot \frac{\sin ROq}{\sin qRO} = \frac{OQ \cdot Rq}{RQ \cdot Oq}.$$

$$\text{In the limit } \mu = \frac{OQ \cdot AF}{AQ \cdot OF},$$

$$\text{or, } \quad \mu . AQ . OF = OQ . AF;$$

$$\text{i.e. } \quad \mu q (p - r) = p (q - r),$$

$$\mu \left(\frac{1}{r} - \frac{1}{p}\right) = \frac{1}{r} - \frac{1}{q},$$

$$\therefore \frac{1}{q} - \frac{\mu}{p} = -\frac{\mu - 1}{r},$$

a formula which determines the position of F with respect to O.

Obs. The result of this article is very convenient in the case of refraction through a sphere (see Art. 105).

30. *To trace the relative change of position of the conjugate foci of a refracted pencil.*

Suppose the refracting surface concave, then the formula connecting the positions of the conjugate foci is

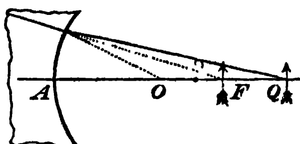
$$\frac{\mu}{v} - \frac{1}{u} = \frac{\mu - 1}{r},$$

where the lines v, u, r are supposed positive to the *right*. It is clear from the formula that if u *increases* or *decreases*, v must also *increase* or *decrease*, i.e. Q, F always move in the *same direction*.

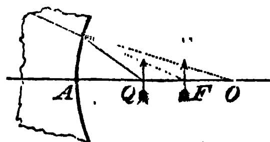
(i) Suppose $u = \infty$, i.e. the incident rays parallel, then

$$v = \frac{\mu}{\mu - 1} r,$$

and F coincides with F_1 the *principal focus*.



(ii) As Q moves from an infinite distance up to O, F also moves up to O, and Q, F will coincide at O. Since when $u = r$, v also $= r$.

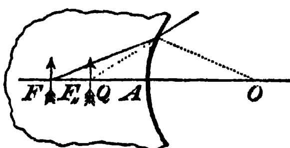


(iii) As Q moves from O up to A, F also moves from O up to A, and Q, F coincide at A.

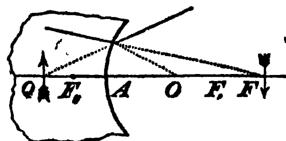
(iv) When u becomes negative, or the incident pencil convergent, let $F_{\prime\prime}$ be a point such that

$$AF_{\prime\prime} = \frac{r}{\mu - 1};$$

then whilst Q moves from A to $F_{\prime\prime}$, F will move from A to $-\infty$.



(v) And lastly, when AQ becomes $> \frac{r}{\mu - 1}$, v becomes positive, and whilst Q moves from $F_{\prime\prime}$ to $-\infty$, F will move from $+\infty$ to F_1.



Similarly, the change of relative position may be traced when the refracting surface is convex.

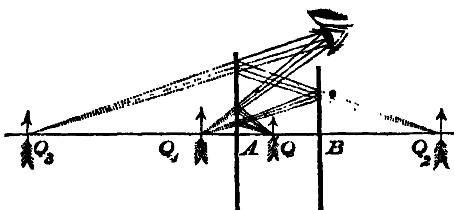
The *remarks* of Art. 26 apply also to this case.

31. The reflecting or refracting surfaces have been considered spherical in the preceding articles, because such are the surfaces which generally occur in the construction of optical instruments. When the surface is any other figure of revolution, it is in general sufficient, when the pencil is not very large, to consider the surface the same as the spherical surface generated by the circle of curvature at the vertex of the generating curve.

The case of a reflecting paraboloid of revolution is however of some interest, inasmuch as its properties are sometimes employed in large telescopes, like that of Lord Rosse.

Illustrations and Examples.

32. *A luminous point being placed between two parallel plane mirrors, to find the position of the images formed by successive reflexions at the mirrors.*



Through the luminous point Q let the line AQB be drawn perpendicular to the two mirrors A, B and produced indefinitely both ways.

Then taking $AQ_1 = AQ$, Q_1 will be the geometrical focus or image of rays emanating from Q and reflected at the mirror A. The rays so reflected from A and diverging from Q_1 will be incident on the second mirror B, and if we take $BQ_2 = BQ_1$, Q_2 will be the focus of the rays after reflexion

from B. The rays will then be incident upon A, diverging apparently from Q_2 and be reflected at A, from a point Q_3 such that $AQ_3 = AQ_2$, and so on.

Again, the rays diverging from Q and incident on the second mirror B will have a geometrical focus or image in Q', where $BQ = BQ'$; the rays diverging from this point Q' and incident on the first mirror A, will after reflexion at A diverge from a point Q'' such that $AQ'' = AQ'$, and so on. Thus there are two sets of images, each infinite in number, all arranged on the line AB and becoming more and more distant after each reflexion. (The second set are not put in the figure to avoid confusion.)

The distances QQ_1, QQ_2... may be easily calculated. For putting $QA = a$, $QB = b$, $AB = a + b = c$, we have

$$QQ_1 = 2AQ = 2a,$$

$$QQ_2 = BQ + BQ_1 = QQ_1 + 2BQ = 2a + 2b = 2c,$$

$$QQ_3 = AQ + AQ_2 = QQ_2 + 2AQ = 2c + 2a,$$

$$QQ_4 = \ldots = 4c.$$

And generally,

$$QQ_{2n} = 2nc,$$

$$QQ_{2n+1} = 2nc + 2a.$$

Similarly, for the second set of images, we should have

$$QQ' = 2b,\ QQ'' = 2c,\ QQ''' = 2c + 2b,\ QQ^{\text{iv}} = 4c,$$

$$QQ^{(2n)} = 2nc,\ QQ^{(2n+1)} = 2nc + 2b.$$

If Q is midway between A and B, $a = b$, and the two sets will form a series which are successively at equal distances ($= c$) from each other.

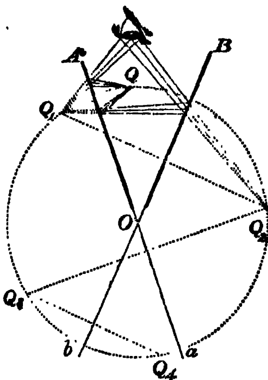
Obs. The diagram will be sufficient to indicate the manner in which the images are seen by an eye in a given position, the small pencil received by the eye passing be-

tween the reflexions in the same manner as if it diverged from the successive images.

A familiar illustration of the above may be noticed in a drawing-room where there are two mirrors on opposite walls, parallel to each other,—an interminable series of images of the objects in the room is observed.

33. *A luminous point being placed between two plane mirrors inclined to one another at a given angle, to find the position and number of the images formed by successive reflexions at the mirrors.*

Let the figure represent a section of the mirrors A, B by a plane passing through the luminous point Q, and perpendicular to each mirror, and therefore perpendicular to their line of intersection—which is represented by the point O.



Describe a circle with centre O and passing through Q. Draw QQ_1 perpendicular to the first mirror A, and meeting the circle in Q_1. Then Q, Q_1 are equidistant from the mirror A, and Q_1 is the *geometrical focus* of rays diverging from Q and reflected at the mirror A, i.e. Q_1 is the image of Q formed by reflexion at A. Draw Q_1Q_2 perpendicular to the second mirror B, then in like manner Q_2 is the image formed by rays reflected from B, after reflexion at A; the course of the rays between the two reflexions at A, B being the same as if they diverged from Q_1. Similarly drawing Q_2Q_3 perpendicular to A or A produced, we obtain a third image Q_3 formed by rays reflected at A, B, and again at A, and so on. Thus we get a set of images formed by rays *first* reflected at A, then at B, again at A, and so on in succession.

In like manner we should get a second set of images formed by rays *first* reflected at B, then at A, again at B, and so on in succession. To avoid confusion we have not marked this second set of images in the figure.

To determine the position of these images, let arc $AQ = \alpha = \angle QOA$, $\angle QOB = \beta$, $\angle AOB = \delta = \alpha + \beta$.

Then, $QQ_1 = 2QA = 2\alpha$,

$$QQ_2 = BQ + BQ_1 = QQ_1 + 2BQ = 2\alpha + 2\beta = 2\delta,$$
$$QQ_3 = AQ + AQ_2 = QQ_2 + 2AQ = \quad 2\delta + 2\alpha,$$
......=..

And in general in the first set of images

$$QQ_{2n} = 2n\delta, \quad QQ_{2n+1} = 2n\delta + 2\alpha.$$

Similarly in the second set of images Q', Q''...

$$QQ^{(2n)} = 2n\delta, \quad QQ^{(2n+1)} = 2n\delta + 2\beta.$$

The number of images is in this case limited; for when we arrive at an image within the $\angle aOb$,—i.e. the $\angle AOB$ produced backward,—the rays which proceed after any reflexion as if they emanated from this image—(Q_4 *in the figure*),—can never fall upon either mirror again, since their directions have already intersected the plane of each mirror,—and consequently cannot be again reflected.

(i) Suppose Q_{2n} the first image which falls within arc ab, then $QQ_{2n} = 2n\delta$ must be not < arc QBa, but < arc QBb; i.e.

$$2n\delta \text{ not} < \pi - \alpha, \text{ but} < \pi + \beta.$$

(ii) Suppose Q_{2n+1} the first image which falls within arc ab, then $QQ_{2n+1} = 2n\delta + 2\alpha$ must be not < arc QAb, but < arc QAa; i.e.

$$2n\delta + 2\alpha \text{ not} < \pi - \beta, \text{ but} < \pi + \alpha,$$

or remembering that $\alpha + \beta = \delta$,

$$(2n+1)\,\delta \text{ not} < \pi - \alpha, \text{ but} < \pi + \beta.$$

These limits are the same, so that if the r^{th} is the first image which falls within ab, or the *last* of the first set of images, we have

$$r \text{ not} < \frac{\pi - \alpha}{\delta}, \text{ but} < \frac{\pi + \beta}{\delta};$$

i.e. r is the integer which lies between $\frac{\pi-\alpha}{\delta}$ and $\frac{\pi+\beta}{\delta}$, or if $\frac{\pi-\alpha}{\delta}$ happens to be an integer, r is equal to this integer; and we may remark that as the difference of these limits is equal to unity, there is only one value of r, and therefore no ambiguity.

COR. 1. There will also be a number r' of images in the second set, and therefore $r+r'$ images in all: r and r' will either be equal, or differ by unity at most.

Suppose $\alpha > \beta$, then $\frac{\pi-\alpha}{\delta}, \frac{\pi-\beta}{\delta}, \frac{\pi+\beta}{\delta}, \frac{\pi+\alpha}{\delta}$ are in order of magnitude, and if there is no integer between $\frac{\pi-\beta}{\delta}$ and $\frac{\pi+\beta}{\delta}$, then $r \sim r' = 1$, but if there is an integer between $\frac{\pi-\beta}{\delta}$ and $\frac{\pi+\beta}{\delta}$, then $r = r'$.

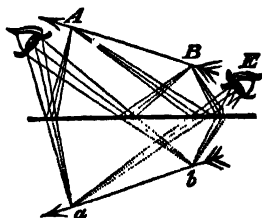
COR. 2. If δ be an exact submultiple of π, then will $r = \frac{\pi}{\delta}$, and the last images of each set will coincide,—and the total number of images will be $2r-1$.

A simple and well-known illustration of this last case (i.e. Cor. 2), is afforded by the *Debuscope*, which consists of two small plane mirrors inclined at 90° to each other, so that $\delta = \frac{\pi}{2}$ and therefore $r = 2$. Hence of any object placed within the quadrant included by the mirrors, the instrument presents three images symmetrically arranged in the three remaining quadrants.

As an illustration of the general case, the student may discuss the following.

Example. *A luminous point moves about between two plane mirrors which are inclined at an* $\angle$ 27°. *Prove that at any moment the number of images of the point is* 14 *or* 13 *according as the angular distance of the point from the nearer mirror is greater or not greater than* 9°.

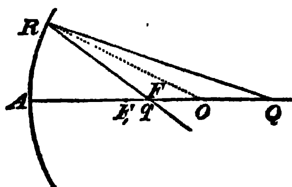
34. When an object as AB is viewed by reflexion at a plane mirror by an eye in any position, the image is seen by pencils which enter the eye as if they proceeded directly from the image, and this image will appear in the same position whatever be the position of the eye; since all the rays proceeding from any point of the object are reflected accurately from a point.



But there is a curious perversion with respect to right and left of the relative position of the parts of the object and image. Thus when a person views himself in a looking-glass, the image of his right hand is apparently what would be the left hand of a man standing in the position of the image. This is not much remarked in consequence of the general symmetry of the right and left sides of a man. As a simple illustration of this perversion, observe the image of a page of a book held before the mirror: or again, write a few words on a sheet of paper, take off the superfluous ink on a sheet of blotting paper, and hold the blotting paper in front of the mirror, when the original writing will be seen in its original order.

35. The relation $\frac{1}{v}+\frac{1}{u}=\frac{2}{r}$ (Art. 21) which connects the conjugate foci of a reflected pencil expresses the law of reflexion more directly than may at first sight be supposed.

Since AR $(=y)$ ultimately vanishes, it is *very small* compared with the other lines in the figure, u, v, r. Hence we may (neglecting cubes of small quantities and therefore putting $\sin\theta=\theta$)



write $\angle RqA=\frac{y}{v}$, $\angle ROA=\frac{y}{r}$,

$\angle RQA=\frac{y}{u}$,

and since $\angle QRO=\angle ORq$,

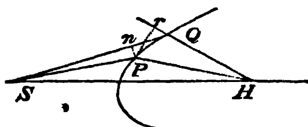
$$\therefore \angle RqA - \angle ROq = \angle qRO = \angle ORQ$$
$$= \angle ROA - \angle RQA;$$
$$\therefore \frac{y}{v} - \frac{y}{r} = \frac{y}{r} - \frac{y}{u}, \text{ or } \frac{1}{v} - \frac{1}{r} = \frac{1}{r} - \frac{1}{u},$$
$$\text{or } \frac{1}{v} + \frac{1}{u} = \frac{2}{r}.$$

This equation which is approximately true when y is small, becomes strictly true in the limit when $y = 0$.

In a similar manner it may be shewn that the relation $\frac{\mu}{v} - \frac{1}{u} = \frac{\mu - 1}{r}$ (Art. 27), when put in the form $\frac{y}{r} - \frac{y}{u} = \mu\left(\frac{y}{r} - \frac{y}{v}\right)$, expresses, simply the law of refraction, $\sin\phi = \mu \sin\phi'$, or $\phi = \mu\phi'$ when the angles ϕ, ϕ' are taken indefinitely small.

36. *Rays diverge from a point S; to find a surface which will refract them accurately to another given point H.*

The surface will be one of revolution about the line joining the two points S, H of divergence and convergence, as axis.



Let SP, SQ be two contiguous rays diverging from S and refracted at P, Q accurately to the point H, in one plane passing through S, H,—μ the refractive index from the first medium into the second. Draw Pn perpendicular to SQ, and Pr perpendicular to HQ produced.

Then we may regard Qn as the *increment* of SP, and Qr as the decrement of HP, in passing from P to Q; one of the two lines SP, PH being increased and the other diminished, since the normal at P must fall between S and H. ϕ, ϕ' the angles of incidence and refraction at P, and PQ so small as to be regarded as coincident with its chord.

Then
$$\delta . SP = Qn = PQ . \sin\phi,$$
$$\delta . HP = -Qr = -PQ . \sin\phi';$$
$$\therefore \delta . SP + \mu . \delta . HP = PQ(\sin\phi - \mu\sin\phi') = 0,$$

in the limit when Q moves up to P.

Hence, integrating

$$SP + \mu \,.\, HP = C \text{ a constant} \quad \text{..............(i).}$$

This equation is that of a curve which by revolution about SH generates a surface such as is required; and which is commonly called an *aplanatic* surface.

COR. 1. If the rays diverging from S, are to *diverge* from a point H after refraction, the curve required would be

$$SP - \mu \,.\, HP = C \quad \text{.................(ii).}$$

COR. 2. If the surface is required which will *reflect* rays proceeding from S, accurately *to* or *from* H, we should obtain for the forms of the generating curves $SP + HP = C$, and $SP - HP = C$ respectively, by proceeding in a manner similar to the above.

These results can of course be obtained independently by direct investigation. They represent respectively an ellipse and a hyperbola; and it may be easily shewn *inversely* that rays diverging from one focus of an ellipse converge to the other focus after reflexion at the curve, and that rays diverging from one focus of a hyperbola will after reflexion at the curve diverge from the other focus.

A particular case of either of these, is a paraboloid of revolution in which rays incident on the concave side of the surface and parallel to the axis will be reflected to the focus, and *vice versâ*. This property of a paraboloid has been employed in forming specula for reflecting telescopes.

COR. 3. If the equation $SP - \mu \,.\, HP = C$ were expressed in rectangular co-ordinates, it would give an algebraic curve of the fourth degree. In the particular case of $C = 0$, the surface will be a sphere.

The discussion of *aplanatic* surfaces was originally pursued by Newton and Descartes,—in fact the class of curves included in the equation (i) are frequently called *Cartesian ovals;* but with the exception of paraboloids referred to in the previous corollary, they are now become questions of curiosity.

See Newton's *Principia*, Bk. I. § 14, Prop. 97.

CHAPTER II.

ILLUMINATION OF SURFACES.

37. WHEN a pencil of light emanates from a luminous point and is propagated in a uniform medium if we suppose its intensity unaltered by the absorption of any portion of it by the medium, yet from other causes the illumination at any point of a surface exposed to the light is different in different positions and at different distances from the surface.

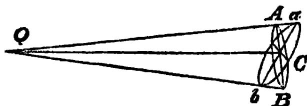
38. *Def.* The illumination at any point of a surface exposed to light is measured by a quantity I: $I\kappa$ being the amount of illumination of an indefinitely small area κ of the surface contiguous to the point in question,—some standard degree of illumination being referred to as a unit.

Hence, if the same quantity of light fall on two very small areas,—$I\kappa$ being the same for each,—the illuminations at any point of these areas are inversely as the areas.

39. *When a small plane area is illuminated by a pencil of rays emanating from a point, the illumination at any point of the area*

$$\propto \frac{\textit{cosine of angle of incidence}}{(\textit{distance from origin})^2}.$$

Let QAB be a small conical pencil of light from an origin Q, ACB, aCb a circular and oblique section of it through a point C in the axis of the pencil.



If aCb be a small plane area illuminated by the pencil,

(i) In all sections parallel to aCb, the quantity of light in the pencil being the same, the illumination at a point is inversely as the area of the section (Art. 38); i.e. inversely as $(\text{dist.})^2$ from Q.

(ii) In all sections through C at different inclinations to the axis, the quantity of light received being the same as that received by ACB, the illumination varies as the area inversely. But if the pencil be supposed so small that ACB may be regarded as the orthogonal projection of aCb,

$$\text{area } aCb = \frac{\text{area } ACB}{\cos \angle ACa},$$

$$\therefore \text{ illumination in } aCb = \text{illumination in } ACB \,.\, \cos \angle ACa$$
$$\propto \cos \angle ACa,$$

and $\angle ACa$ being the inclination of the planes ACB, aCb, is the angle between the perpendiculars to these planes at C, *or* is the angle of incidence of QC.

Hence, when both the distance and angle of incidence vary together,—the illumination at any point of the area

$$\propto \frac{\text{cosine angle of incidence}}{(\text{distance})^2}.$$

COR. The illumination at any point of the area

$$= C \,.\, \frac{\text{cosine } \angle \text{ of incidence}}{(\text{distance})^2},$$

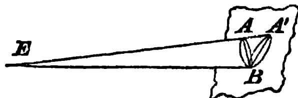
where C depends only on the brightness of the illuminating point, and is the illumination at any point of a small area directly exposed to the pencil at a distance *unity* from the origin.

40. *Remark.* The preceding result will apply to a curved surface illuminated by a pencil of light, since we may regard a *very small* portion of the curved surface contiguous to a given point on it, as coincident appreciably with the tangent plane to the surface at the point considered:—we may also regard the *intrinsic brightness* of any luminous surface as proportional to the amount of light emitted by *a unit of area normal to the surface.*

41. It is found by experiment that luminous surfaces appear to be of the same brightness at any point, whatever be the inclination of the surface at that point to the axis of the pencil by which it is seen. The Sun's disc, *for example,* is equally bright at all distances from the centre. The apparent

intrinsic brightness of a bar of red-hot iron is not sensibly altered by inclining it obliquely to the eye.

Thus if $A'B$ be a *small* portion of a luminous surface viewed *obliquely* by an eye E, the amount of light received from it will be the same as that received from a portion AB viewed *directly*: AB being the section, perpendicular to the axis of the cone, of which E is the vertex and $A'B$ the base; i.e. the intensity of emission of light in direction $A'E$: intensity of emission perpendicular to the surface at A' :: area AB : area $A'B = \sin\theta : 1$, where θ is the angle which EA' makes with the luminous surface at A'.

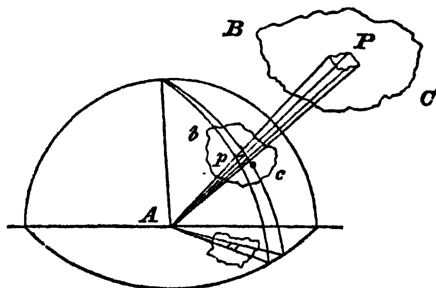


Hence it follows that *the copiousness of emission of light from a luminous surface is proportional to the sine of the angle of emanation from the surface.*

42. The following is a mode of calculating the illumination at any point of a surface illuminated by a given surface of uniform brightness.

Let A be a small plane area illuminated by a surface BC of uniform brightness.

About A as centre describe a sphere, and let a line through A moving round the boundary of BC intersect the surface of this sphere in the curve bc. Take also an element



P of the surface BC, and let p be the corresponding element of the spherical surface formed as before. Let α, θ be the

inclinations of AP to the illuminated plane and to the surface at P.

If the element at P be regarded as an origin of light, illumination at A from it $= C\frac{\sin\alpha}{AP^2}$. area P. $\sin\theta$ (39, 41), and if the surface of the sphere be supposed of the same uniform brightness with BC, and the element at p be considered an origin of light, the illumination at A from it would be

$$= C\frac{\sin\alpha}{Ap^2} \text{ area } p.$$

But $$\frac{\text{area } P . \sin\theta}{AP^2} = \frac{\text{area } p}{Ap^2}$$

i.e. the illumination at A from corresponding *elements* of BC and bc would be equal, and therefore the illumination from BC would be the same as that from bc.

But the surface of the sphere at p being inclined to that of A at an $\angle = \frac{\pi}{2} - \alpha$, it follows that

area p . $\sin\alpha$ = area of projection of p on the plane of A;

$\therefore$ illumination at A from p

$= \frac{C}{Ap^2}$. area of projection of p on the plane of A;

$\therefore$ illumination at A from BC = illumination at A from bc

$= \frac{C}{Ap^2}$. area of projection of bc on the plane of A.

If therefore the area of the projection of bc can be found the illumination at A may be known.

Ex. If A be illuminated by a sky equally bright in all directions above the horizon,

area of projection of $bc = \pi . Ap^2$;

$\therefore$ illumination at $A = \pi C$.

43. *Remark.* If the illuminated surface be *curved,* then the small area A must be regarded as a small element of the tangent plane to the illuminated surface at A.

The student who is acquainted with *solid geometry* will have little difficulty in applying the above process by means of the usual systems of co-ordinates to any case that may occur.

If the illuminating surface BC be not of uniform brightness, the quantity C which occurs in the above investigation will not be constant but will be a function of the position of P on the surface BC,—and the resulting illumination at A will in general have to be determined by integration.

44. *Objects appear equally bright at all distances.*

The apparent intrinsic brightness of an object may be measured by the amount of light received from it by the eye, divided by the area of the picture on the retina of the eye. But this area is proportional to the apparent superficial magnitude of the object; i.e. to its real area A divided by the square of its distance D, or $\propto \frac{A}{D^2}$,—moreover the apparent light or illumination is $\frac{A \cdot C}{D^2}$, where C is the real intrinsic brightness. Consequently the apparent intrinsic brightness

$$\propto \frac{A \cdot C}{D^2} \div \frac{A}{D^2} \propto C,$$

which is not dependent on A or D. The apparent intrinsic brightness is therefore the same at all distances, and is proportional to the real intrinsic brightness of the object.

This conclusion is usually expressed by saying that *objects appear equally bright at all distances,* which must be understood only of apparent intrinsic brightness, and the truth of which supposes no loss of light by absorption to take place in the media traversed.

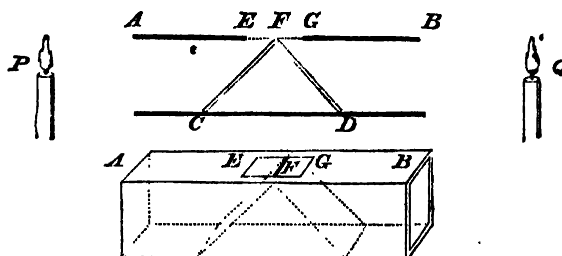
44*. To compare the intrinsic brightness or illuminating power of two sources of light at moderate distances, we may cause the rays emitted by each to fall separately on two

screens physically identical in character in a direction normal to the surfaces or pretty nearly so: the distances may then be varied until the illuminations are sensibly equal.

Thus if A be the area of the projection of one luminous surface (α) on a plane perpendicular to the direction of the luminous rays, D the distance of α from the screen, C the intrinsic brightness of α, the illumination on the screen may be measured by $\frac{CA}{D^2}$; if C', A', D' be similar quantities for another source of light β and if the distances be varied until the illuminations from the two sources are equal we shall have $\frac{CA}{D^2} = \frac{C'A'}{D'^2}$, which will enable us to determine the ratio of $C : C'$.

45. Various instruments have been constructed for comparing the illuminating powers of two sources of light: it will suffice to describe two of them.

Ritchie's Photometer consists of a rectangular box, about two inches square, open at both ends, of which $ABCD$ is a



section. The inner surface is blackened so as to absorb extraneous light. Within the box inclined at angles of 45° to its axis are placed two rectangular pieces of looking-glass, FC, FD, *cut from one and the same rectangular strip* to ensure the exact equality of their reflecting powers, and fastened so as to meet at F, in the middle of a narrow slit EFG about an inch long and one-eighth of an inch broad, which is

covered with a slip of fine tissue or oiled paper. The rectangular slit should have a slip of blackened card at F, to prevent the lights reflected from the looking-glasses mingling with each other.

When we wish to compare the illuminating powers of two sources of light—(two flames for instance)—P and Q, they must be placed at such a distance from each other, and from the instrument between them, that the light from every part of each shall fall on the reflector next it,—and be reflected to the corresponding portion of the paper EF or FG. The instrument is then to be moved nearer to one or the other, till the paper on either side of the division F appears equally illuminated. To judge of this it should be viewed through a tube blackened within, one end resting upon the upper part AB of the photometer, the other applied quite close to the eye. When the lights are thus exactly equalized, it is clear that the total illuminating powers of the luminaries are directly as the squares of their distances from the middle of the instrument.

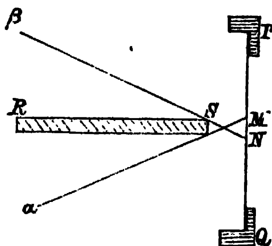
Note. To render the comparison of the lights more exact, the equalization of the lights should be performed several times, turning the instrument end for end each time. The mean of the several determinations will then be very near the truth.

By means of the above instrument we may obtain an easy experimental proof of the decrease of light as the inverse square of the distance. For if we place four candles at P and one at Q (as nearly equal as possible and burning with equal flame) it is found that the portions of the paper EF, GF will be equally illuminated when the distances PF, QF are as 2 : 1, and so for any number of candles at each side.

45*. In *Foucault's Photometer*, the two sources of light (α, β)—gas lights for instance—which are to be compared, act separately on two different parts of a vertical plate of porcelain PQ sufficiently thin to be translucent. The opaque vertical screen RS, which separates the two illuminations one from the other, can be made to approach or retire from PQ at pleasure. If such a position be given to it that the vertical planes

3—2

drawn through αM, βN which bound the portions illuminated severally by α and β, intersect on the porcelain plate or nearly so—the band MN illuminated by both α and β can be made as narrow as we please—and if the distances of α and β be varied until the illuminations of the two parts of the plate PM, QN are sensibly equal, the intrinsic intensities of α, β can be compared.

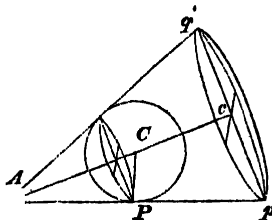


Instead of the plate of porcelain—any uniform translucent membrane may be used:—or a plate of ground-glass, or of glass covered with a thin film of matter.

Note. The student may consult Deschanel's *Natural Philosophy*—Professor Everett's edition 1882, Art. 1036, &c., on *Measure of Brightness.*

46. A plane surface touches a self-luminous sphere: to find the illumination of the surface at any point.

Let A be any point in the plane surface which the sphere whose centre is C touches at P. Join CP. With centre A describe a spherical surface, and let a straight line through A moving round the sphere C so as to define the portion of it from which A receives illumination, intersect the spherical surface described about A in the small circle pq, the plane of which meets AC or AC produced in c.



Then projection on plane AP, of the curved surface pq,

$$= \text{projection of the plane circle } pq$$

$$= \pi \, cp^2 \, . \, \frac{cp}{Ap}. \quad \text{(Hymers, \textit{Three Dim.} Art. 81.)}$$

Therefore illumination at A from the sphere (Art. 42)

$$=\frac{C}{Ap^2}\cdot\frac{\pi\cdot cp^2}{Ap}=\pi\cdot C\cdot\left(\frac{cp}{Ap}\right)^3=\pi C\cdot\left(\frac{CP}{AC}\right)^3\cdot\propto\frac{1}{AC^3}.$$

47. Light is propagated with finite velocity.

The eclipses of Jupiter's satellites are observed to happen sooner when he is in *geocentric opposition* (and consequently nearest to the Earth), and later when he is in *conjunction* (or farthest from the Earth), than they ought to happen according to calculations made on supposition that he is at his mean distance from the Earth. The difference is accounted for by the hypothesis that light requires a finite time for its transmission. This supposition is confirmed by its explaining satisfactorily the apparent displacements of heavenly bodies, called *aberration*.

The *coefficient of aberration* being the same for heavenly bodies at different distances, it appears that the velocity of light is uniform in the same medium. From observations on the aberration of stars, this velocity was supposed to be about 192000 miles per second in vacuum, *or rather* in that space which intervenes between us and the planets and fixed stars. Thus the time required for light to travel from the Sun to the Earth is about 8 minutes. There is reason to think however that this value for the velocity is too great: *see next article.*

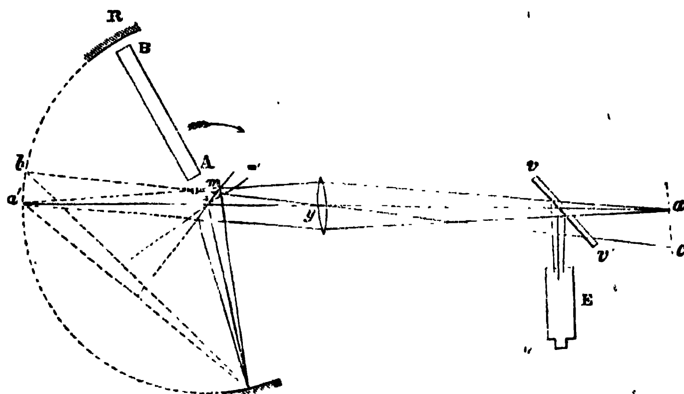
The discrepancies above referred to with respect to the eclipses of Jupiter's satellites were first noticed about A.D. 1675; but there is some uncertainty whether *Roemer* or *J. D. Cassini* first observed them. The finite velocity of light was employed by *Bradley* to explain the aberration of the stars, A.D. 1728.

M. Fizeau was the first person who rendered the velocity of light sensible in experiments made on the surface of the Earth (see *Comptes Rendus*, t. 29, pp. 90, 132, July 23, 1849). A few months afterwards, the principle of a rotating mirror, which had been used by *Prof. Wheatstone* in measuring the velocity of electricity, was employed both by M. Fizeau and M. Foucault in experiments described by them in memoirs

presented on the same day to the Academy of Sciences (*Comptes Rendus*, t. 30, *May* 6, 1850). A subsequent memoir by *M. Fizeau* is given in the *Comptes Rendus*, t. 33.

47*. We proceed to give a short account of *Foucault's experiment.*

A beam of sunlight is transmitted by means of a reflector into a dark room through a small square aperture in the window shutter.

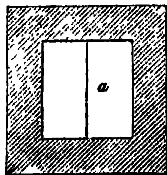


zm is a small plane mirror capable of revolving about an axis perpendicular to the plane of the paper in the direction of the arrow; *z* being the centre of a spherical mirror $a''\,a'$ (of which only a small portion about a'' is a reflector, subtending about 7° at *z*); *zy* the axis of a lens at *y* whose focal length *f* is such that a pencil of rays proceeding from a point *a* in the aperture after refraction through the lens *y* would come to a focus at a' on the spherical mirror, but being reflected by the plane mirror *zm* actually comes to a focus at a'' on the spherical mirror—the line $a'\,a''$ being perpendicular to the plane *zm*: this pencil being reflected at a'' pursues the same course reversed, so that after being again reflected by the plane mirror *zm* and refracted by the lens, comes to a focus again at *a*; so that this arrangement of lens and reflectors produces an image of an object *a* coinciding with the object itself.

As the mirror *zm* is made to revolve, whenever its position during a revolution is such as to cause the pencil from *a* to fall upon a'' and be there reflected, the image of *a* will appear,—and for other positions of *zm* it will disappear. If the mirror *zm* do not make more than about 30 revolutions a second the intermittent appearances are distinct; but when the number exceeds 30, the image appears steady and persistent—and coincident with *a*, so long as the plane mirror *zm* revolves with such a moderate velocity that its change of position is insensible in the short interval which is occupied by light in passing from *z* to a'' and back again; but if the *zm* is made to revolve rapidly (*several hundred times a second*) this is no longer the case, but the rays which were reflected from the plane mirror when in the position *zm*, after passing to a'' and being there reflected, are again incident on the plane mirror when it occupies a position zm':—the $\angle\, mzm'$ $(=x)$ being the angle through which *zm* has turned in the time (2τ) which has been occupied by light in passing from *z* to a'' and back again to *z*.

If we draw $a''\,b'$ perpendicular to $m'z$, the pencil after reflexion at zm' and refraction through the lens will come to a focus at a point *c* in $b'y$ produced, at a distance from *y* equal to *ya*;—that is, *c* the image of *a* is deflected through a small but sensible space in a direction perpendicular to the axis of the lens and also perpendicular to the axis of rotation of the plane mirror.

For convenience of observation, a piece of plate-glass vv' which allows the direct rays from *a* to pass through it without obstruction is interposed at an $\angle\, 45^{\circ}$ to *ay*—and the returning rays are partially reflected at its front surface and viewed by an eye-piece *E*. The object *a* is a platinum wire drawn across the centre of the square aperture in the shutter, and the position of the eye-piece *E* is adjusted so that the image of *a* formed by the apparatus when the mirror *zm* is quiescent coincides with the wire fixed in the principal focus of the eye-piece:—



when the mirror *zm* is made to revolve rapidly, this image is seen displaced parallel to itself through a space equal to *ac*.

Calculation.

Let $\epsilon = ac$ = displacement of the wire,

$r = za''$ = radius of the spherical mirror,

n = number of revolutions of plane mirror in 1",

θ, ϕ the angles $a'zb'$, $a'yb'$ in circular measure, $b = ay$, $l = yz$, v = velocity of light, $\therefore r + l = ya'$; then we shall have

$$v\tau = r, \frac{x}{2\pi n} = 2\tau = \therefore \frac{2r}{v},$$

whence $x = \dfrac{4\pi nr}{v}$.

Also $\theta = \angle a'zb = 2 \angle a'a''b = 2 \angle mzm' = 2x = \dfrac{8\pi nr}{v}$

and $$\text{arc } a'b = r\theta = (r + l)\phi,$$

$$\therefore \epsilon = b\phi = \frac{br}{r+l}\theta = \frac{8\pi nr^2 b}{v(r+l)},$$

or $v = \dfrac{8\pi nr^2 b}{\epsilon(r+l)}$ which expresses the velocity of light in terms of quantities which can be measured.

In the experiments made by M. Foucault

$r =$ 4 *metres*,

$b =$ 3 ,,

$l =$ 1·1818 ,,

$f =$ 1·9 ,,

and when $n = 800$ it was found that e was = ·0006ᵐ, which numbers would give v = 192540 miles nearly.

From subsequent experiments however made with great care, *M. Foucault* (*Comptes Rendus*, t. 55, pp. 501, 792; *anno* 1862) concludes that v = 298000 *kilometres* = 185172 *miles* per *second*—(a *metre* being = 39·371 *inches*:)—and he estimates this to be the correct value within $\frac{1}{500}$ part of the whole.

A careful repetition of Fizeau's experiment, made in 1874 by M. Alfred Cornu, gave the *velocity of light* to be 186,700 *miles* per *second.* (*Nature,* xi. p. 274.) The Rumford medal was awarded in 1878 by the *Royal Society* of London to M. Cornu for this and other Optical Researches.

Careful experiments by Foucault's method made in 1879 by A. A. Michelson, an Officer of the United States navy, gave the velocity of light to be 186,380 *miles* per *second.* (*Nature,* xxi. p. 226.)

Sir John Herschel says that we are authorized to conclude that in estimating the *velocity of light* at 186,000 *miles* per *second,* we are within a thousand miles of the truth. (*Familiar Lectures,* p. 234.)

By Foucault's apparatus the velocity of light in water can also be determined: for this purpose a tube *AB* filled with distilled water is interposed between the revolving mirror *zm* and another concave mirror *R*, similar to *a″*. The light reflected along this tube by the mirror *zm*, when it is in a suitable position in the course of its revolution, will be reflected back again by *R*, and thus the light will traverse the column of water twice in its course. If the velocity of light in water be less than in air, the time of passage along the tube will be longer than for an equal length of air, and therefore displacement of the image observed at *E* would be greater than when no column of water is interposed,—and such was observed to be the case.

For a complete account of these interesting experiments, the student may consult the volumes of the *Comptes Rendus* above referred to; or *Pouillet, Elements de Physique &c.* Tome 2, 7th edition, 1856; *Jamin, Cours de Physique,* Tome 3, 1869.

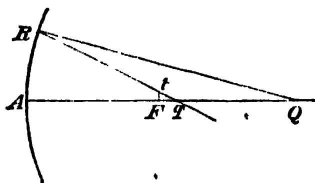
CHAPTER III.

ABERRATION OF SMALL DIRECT PENCILS.

48. It was stated at (18) that the determination of the form of a direct pencil after reflexion or refraction would consist of two parts,

(i) The determination of the geometrical focus,

(ii) The determination of the vertex q of any one of the cones of rays before described.



We will here proceed with the latter of these parts in a few of the more simple cases.

The equation which gives the distance Aq will not in general admit of direct solution, but is solved by successive approximations,—the approximation being conducted according to powers of AR, the half-breadth of the conical shell in question, which in such pencils as occur in the computations of instruments is very small compared with the other lines involved in the equation. It will appear in the course of our calculations that AR^2 is the *lowest* power of AR involved in the equations. The square and higher powers of AR being at first neglected, a *first approximate value* of Aq is obtained. Next, by neglecting the cube and higher powers of AR, and substituting in the coefficient of AR^2, the approximate value of Aq before obtained, a *second approximate value* is obtained. By this means the value of Aq may be determined to any degree of accuracy. It is found sufficient in the calculations which have reference to instruments to carry the approximation as far as AR^2.

49. We premise the following theorem for the purpose of shewing that an approximate value of a quantity to be determined, may be substituted in the small terms of the equation.

Suppose v a quantity whose value is to be found, and which is given implicitly by an equation of the form

$$v = V + f(v) \cdot y^2 \text{ ....} (A),$$

where V involves known quantities only, and is independent of y, the small quantity by powers of which the approximation is conducted,—and $f(v)$ is a function of v and known quantities.

Then by substitution

$$f(v) = f\{V + f(v) \cdot y^2\} = f(V) + f'(V) \cdot f(v) \cdot y^2 + \dots$$

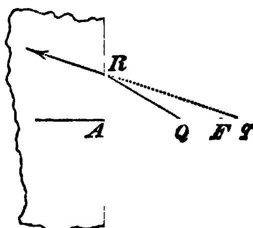
by Taylor's Theorem;

$$\therefore v = V + f(V) \cdot y^2 + f'(V) \cdot f(v) \cdot y^4 + \dots$$
$$= V + f(V) \cdot y^2,$$

if the approximation extend only to the square of y.

It appears therefore that in the term of (A) which involves y^2, we may substitute the value of v obtained by neglecting y^2, and the result will be true to the order of y^2.

50. *When a pencil is incident directly on a plane refracting surface, to find the point where the direction of a given ray after refraction cuts the axis.*



Let Q be the origin of a pencil whose axis QA is incident directly at A on a plane refracting surface, then QA is the axis of the refracted pencil. Let QR be any ray incident at R, and refracted in a direction which cuts the axis in q, the position of which is to be determined.

Let $AQ = u$, $Aq = v'$; lines being considered positive when measured from A in a direction contrary to that of the incident

pencil. Also let $AR = y$, a quantity of which the cube and higher powers may be neglected.

Now RQA, RqA being equal to the angles of incidence and refraction of the ray QR,

$$\sin RQA = \mu \, . \sin RqA\,;$$

$$\therefore Rq = \mu \, . \, RQ,$$

$$\text{or } \sqrt{(v'^2 + y^2)} = \mu \sqrt{(u^2 + y^2)}\,;$$

$$1 + \frac{y^2}{2v'^2}\Big) = \mu u \left(1 + \frac{y^2}{2u^2}\right), \text{ approximately,}$$

$$v' = \mu u + \left(\frac{\mu}{u} - \frac{1}{v'}\right)\frac{y^2}{2}\,.$$

In the coefficient of y^2, for v' we may substitute μu, the first approximate value of v', and the resulting equation will be true to the order of y^2 (Art. 49);

$$\therefore v' = \mu u + \left(\frac{\mu}{u} - \frac{1}{\mu u}\right)\frac{y^2}{2}$$

$$= \mu u + \frac{\mu^2 - 1}{\mu} \, . \, \frac{y^2}{2u}\,.$$

Cor. Hence it appears by comparing this result with that of Art. 19, that the geometrical focus is the approximate position of the point q, when powers of y above the *first* are neglected.

51. When a pencil is directly reflected or refracted at a surface, the aberration of any ray is the distance between the geometrical focus and the point where the direction of that ray after reflexion or refraction cuts the axis (13).

The aberration of a *pencil* is the aberration of the extreme ray in any section of the pencil through its axis.

Hence the *aberration of the ray* $qR = v' - \mu u$

$$= \frac{\mu^2 - 1}{\mu} \cdot \frac{y^2}{2u}.$$

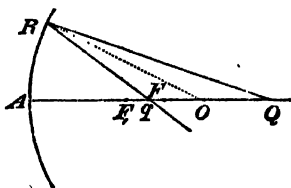
This quantity is positive or negative, i.e. the aberration is *from* or *towards* the refracting surface, according as

$$\mu > 1, \text{ or } < 1,$$

that is, according as the refraction takes place from a *rarer* medium into a *denser* one, or *vice versa.*

52. *When a pencil is incident directly on a spherical reflecting surface, to find the point where the direction of a given ray after reflexion cuts the axis.*

Let Q be the origin of a pencil whose axis QA is incident directly at A on a spherical reflecting surface whose centre is O: then AQ is the axis of the reflected pencil. Let QR be any ray incident at R and reflected in a direction which cuts the axis in q, the position of which is to be determined. Join OR.



Let $AQ = u$, $Aq = v'$, $AO = r$, lines being considered positive when measured from A in a direction opposite to that of the incident pencil. Also let $AR = y$,—a quantity of which the cube and higher powers may be neglected,—on which supposition y may be taken to represent either AR, or the perpendicular from R on AQ, at pleasure.

Then since QRq is bisected by RO;

$$\therefore \frac{QR}{qR} = \frac{QO}{qO}, \text{ or } QR \,.\, qO = qR \,.\, QO.$$

Now $QR^2 = RO^2 + OQ^2 + 2OR \,.\, OQ \,.\, \cos ROA$

$$= r^2 + (u - r)^2 + 2r \,.\, (u - r) \cos \frac{y}{r}$$

$$= u^2 - \frac{u - r}{r} \,.\, y^2,$$

after reduction, since $\cos ROA$

$$= \cos \frac{y}{r} = 1 - \frac{y^2}{2r^2}, \text{ approximately;}$$

$$\therefore\ QR = u \left\{1 - \left(\frac{1}{r} - \frac{1}{u}\right) \frac{y^2}{2u}\right\}.$$

Similarly, $$qR = v' \left\{1 - \left(\frac{1}{r} - \frac{1}{v'}\right) \frac{y^2}{2v'}\right\};$$

$$\therefore\ (r - v')\, u \left\{1 - \left(\frac{1}{r} - \frac{1}{u}\right) \frac{y^2}{2u}\right\} = (u - r)\, v' \left\{1 - \left(\frac{1}{r} - \frac{1}{v'}\right) \frac{y^2}{2v'}\right\}.$$

If each side of this be divided by $u\, v'\, r$

$$\left(\frac{1}{v'} - \frac{1}{r}\right) \left\{1 - \left(\frac{1}{r} - \frac{1}{u}\right) \frac{y^2}{2u}\right\} = \left(\frac{1}{r} - \frac{1}{u}\right) \left\{1 - \left(\frac{1}{r} - \frac{1}{v'}\right) \frac{y^2}{2v'}\right\},$$

or $$\frac{1}{v'} + \frac{1}{u} = \frac{2}{r} + \left(\frac{1}{r} - \frac{1}{u}\right) \left(\frac{1}{v'} - \frac{1}{r}\right) \left(\frac{1}{u} + \frac{1}{v'}\right) \frac{y^2}{2}.$$

If we neglect powers of y higher than the first

$$\frac{1}{v'} + \frac{1}{u} = \frac{2}{r}, \text{ or } v' = v \text{ (Art. 21)},$$

which value we may substitute in the coefficient of y^2, and the resulting equation will be true as far as y^2;

$$\therefore\ \frac{1}{v'} + \frac{1}{u} = \frac{2}{r} + \left(\frac{1}{r} - \frac{1}{u}\right)^2 . \frac{y^2}{r},$$

which gives the position of q.

Cor. 1. $$\frac{1}{v'} - \frac{1}{v} = \left(\frac{1}{r} - \frac{1}{u}\right)^2 . \frac{y^2}{r};$$

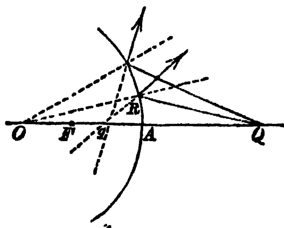
$$\therefore\ v' - v = -\left(\frac{1}{r} - \frac{1}{u}\right)^2 . \frac{y^2}{r} . vv',$$

or putting $v' = v$ in the term involving y^2, we have

$$\textit{aberration of ray } Rq = v' - v = -\left(\frac{1}{r} - \frac{1}{u}\right)^2 \cdot \frac{v^2 y^2}{r}$$

$$= -\frac{\left(\frac{1}{r} - \frac{1}{u}\right)^2}{\left(\frac{2}{r} - \frac{1}{u}\right)^2} \cdot \frac{y^2}{r}.$$

COR. 2. If, as in Art. 24, the position of any ray is determined with reference to the centre O of the surface, we should get



$$\frac{r}{Oq} + \frac{r}{p} = 2 \cos \theta.$$

Write $Oq = q'$, then

$$\frac{r}{q'} + \frac{r}{p} = 2 \cos \theta,$$

$$\text{and } \frac{r}{q} + \frac{r}{p} = 2 \text{ (Art. 24)};$$

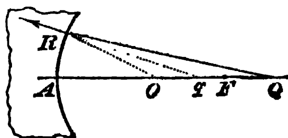
$$\therefore \frac{1}{q} - \frac{1}{q'} = \frac{2}{r}(1 - \cos \theta) = \frac{\theta^2}{r}, \text{ nearly,}$$

$$= \frac{y^2}{r^3}, \text{ nearly}; \because \theta = \frac{y}{r};$$

$$\therefore \text{aberration of ray } qR = q' - q = \frac{y^2 \cdot q^2}{r^3}.$$

53. *When a pencil is incident directly on a spherical refracting surface,—to find the point where the direction of a given ray after refraction cuts the axis.*

Let Q be the origin of a pencil whose axis QA is incident directly at A on a spherical refracting surface whose centre is O; then QA is the axis of the refracted pencil. Let QR be any ray incident at R, and refracted in a direction which cuts the axis in q, the position of which is to be determined. Join OR.



Let $AQ = u$, $Aq = v'$, $AO = r$, lines being considered positive when measured from A in a direction opposite to that of the incident pencil. Also let $AR = y$—a quantity of which the cube and higher powers may be neglected.

$$\text{Then } \mu = \frac{\sin QRO}{\sin qRO} = \frac{\sin QRO \,.\, \sin ROq}{\sin ROQ \,.\, \sin qRO} = \frac{OQ \,.\, Rq}{RQ \,.\, Oq},$$

or
$$\mu \,.\, RQ \,.\, Oq = OQ \,.\, Rq.$$

Now
$$RQ^2 = RO^2 + OQ^2 + 2RO \,.\, OQ \cos ROA$$
$$= r^2 + (u-r)^2 + 2r(u-r)\cos\frac{y}{r}$$
$$= u^2 - (u-r)\frac{y^2}{r}, \text{ nearly};$$

$$\therefore RQ = u\left\{1 - \left(\frac{1}{r} - \frac{1}{u}\right)\frac{y^2}{2u}\right\}.$$

Similarly, $Rq = v'\left\{1 - \left(\frac{1}{r} - \frac{1}{v'}\right)\frac{y^2}{2v'}\right\};$

$$\therefore \mu(v'-r)u\left\{1 - \left(\frac{1}{r} - \frac{1}{u}\right)\frac{y^2}{2u}\right\} = (u-r)v'\left\{1 - \left(\frac{1}{r} - \frac{1}{v'}\right)\frac{y^2}{2v'}\right\};$$

$$\therefore \mu\left(\frac{1}{r} - \frac{1}{v'}\right)\left\{1 - \left(\frac{1}{r} - \frac{1}{u}\right)\frac{y^2}{2u}\right\} = \left(\frac{1}{r} - \frac{1}{u}\right)\left\{1 - \left(\frac{1}{r} - \frac{1}{v'}\right)\frac{y^2}{2v'}\right\},$$

or
$$\frac{\mu}{v'} - \frac{1}{u} = \frac{\mu-1}{r} + \left(\frac{1}{r} - \frac{1}{v'}\right)\left(\frac{1}{r} - \frac{1}{u}\right)\left(\frac{1}{v'} - \frac{\mu}{u}\right)\frac{y^2}{2}.$$

If the square of y be neglected, v the approximate value of v' is given by the equation

$$\frac{\mu}{v} - \frac{1}{u} = \frac{\mu-1}{r};$$

which value of v' may be used in the coefficient of y^2 and the result will be true as far as y^2.

$$\frac{\mu}{v'} - \frac{1}{u} = \frac{\mu-1}{r} + \frac{1}{\mu^2}\cdot\left(\frac{1}{r}-\frac{1}{u}\right)^2\left(\frac{1}{u}+\frac{\mu-1}{r}-\frac{\mu^2}{u}\right)\frac{y^2}{2}$$

$$= \frac{\mu-1}{r} + \frac{\mu-1}{\mu^2}\cdot\left(\frac{1}{r}-\frac{1}{u}\right)^2\left(\frac{1}{r}-\frac{\mu+1}{u}\right)\frac{y^2}{2},$$

which equation determines the position of q.

COR. $$\frac{\mu}{v'} - \frac{\mu}{v} = \frac{\mu-1}{\mu^2}\cdot\left(\frac{1}{r}-\frac{1}{u}\right)^2\left(\frac{1}{r}-\frac{\mu+1}{u}\right)\frac{y^2}{2};$$

therefore the aberration of the ray $Rq' = v' - v$

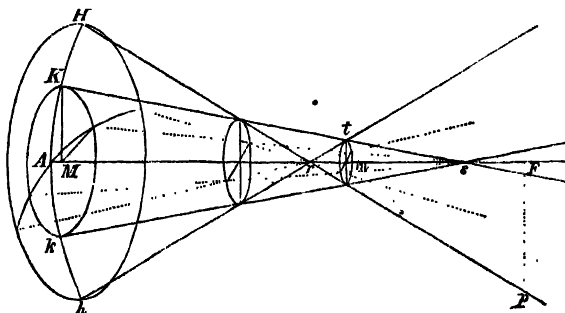
$$= -\frac{\mu-1}{\mu^3}\cdot\left(\frac{1}{r}-\frac{1}{u}\right)^2\left(\frac{1}{r}-\frac{\mu+1}{u}\right)\frac{y^2v^2}{2}$$

$$= -\frac{\mu-1}{\mu}\,\frac{\left(\frac{1}{r}-\frac{1}{u}\right)^2}{\left(\frac{\mu-1}{r}+\frac{1}{u}\right)^2}\left(\frac{1}{r}-\frac{\mu+1}{u}\right)\frac{y^2}{2}.$$

54. *Obs.* In the preceding Articles, since powers of y above the second have been neglected, it is immaterial whether we take for y the length of the arc AR, or the perpendicular distance of R from the axis of the surface.

Least Circle of Aberration.

55. Let AF be the axis of a pencil of rays reflected or refracted directly at a spherical surface; Hr, hr the extreme rays in any plane section of the pencil through the axis,



which meet in the point r of the axis; Ks, ks any other two rays in the same section meeting in the point s of the axis. Suppose hr (produced if necessary) to cut Ks in t and draw tm perpendicular to AF.

Now (i) considering rays incident at different points along AH, when K coincides either with A or H, tm is $= 0$; therefore for some position of K in AH, tm is a maximum.

(ii) If K have the position for which tm is a maximum, a circle with centre m and radius mt, in a plane perpendicular to AF, is the smallest space through which the whole pencil passes. For a circular section of the pencil *to the left* of m is larger in consequence of the converging cone Ksk, and a circular section *to the right* of m is larger in consequence of the diverging cone trp. Such a circle is called the *Least Circle of Aberration* of the directly reflected or refracted pencil.

Obs. In the preceding figure, the aberration is supposed to be *towards* the surface; the student will have little difficulty in drawing a corresponding figure for the case when the aberration is *from* the surface.

In the following calculation of the position and dimensions of this circle, the cube and higher powers of the half-breadth of the pencil will be neglected, as has been done in former cases.

56. *To calculate the position and dimensions of the Least Circle of Aberration after direct reflexion or refraction, at a plane or spherical surface.*

Let $AK = y'$, $AH = y$, $r =$ rad. of spherical surface AK (fig. Art. 55). Draw KM perpendicular to AF.

$$\therefore KM = r \sin \frac{y'}{r} = y',$$

$$AM = r \text{ vers. } \frac{y'}{r} = \frac{y'^2}{2r}, \text{ nearly.}$$

By similar triangles

$$\frac{ms}{mt} :: \frac{Ms}{MK} : \frac{As - \frac{y'^2}{2r}}{y'} = \frac{As}{y'} - \frac{y'}{2r}$$

$$ms = mt \cdot \left(\frac{As}{y'} - \frac{y'}{2r}\right).$$

So $$\frac{mr}{mt} = \frac{Mr}{AH} = \frac{Ar}{y} - \frac{y}{2r};$$

$$\therefore\ mr = mt\left(\frac{Ar}{y} - \frac{y}{2r}\right)$$

Now since mt, depending on the aberration, is at least of the order y'^2,—therefore $y' \cdot mt$ and $y \cdot mt$ may be neglected. Also the difference between Ar and As being the difference of the aberrations of the two rays Hr, Ks, is of the order y'^2; and therefore when multiplied by mt we may regard Ar and As as equal, since the error introduced by so doing is of a higher order than y'^3.

Hence we may take

$$rs = rm + ms = mt \cdot Ar\left(\frac{1}{y} + \frac{1}{y'}\right)$$

But (Art. 52, 53) the aberration of any ray in the same pencil $\propto y^2$;

$$\therefore\ Fs : Fr = y'^2 : y^2.$$

$$\therefore\ rs : Fr = y^2 - y'^2 : y^2;$$

$$\therefore\ Fr \cdot \frac{y^2 - y'^2}{y^2} = rs = mt \cdot Ar \cdot \frac{y + y'}{yy'};$$

$$\therefore\ mt = \frac{Fr}{Ar} \cdot \frac{(y - y')\,y'}{y}$$

Now mt is variable by the change of y', and this value of mt will therefore be a maximum when

$$y - 2y' = 0,\ \text{i.e. } y' = \frac{y}{2}.$$

$$\therefore\ mt = \frac{1}{4} \cdot \frac{Fr}{Ar} y \ldots\ldots\ldots\ldots\ (\text{i}),$$

$$rm = \frac{1}{4} \cdot Fr \ldots\ldots\ldots\ldots\ldots\ (\text{ii});$$

which two equations define the position and magnitude of the *least circle of aberration.*

In other words, the distance of the least circle of aberration from the geometrical focus is three-fourths of the *longitudinal* aberration of the pencil, and its radius is one-fourth of Fp, the *lateral* aberration of the extreme ray.

CHAPTER IV.

FOCAL LINES OF SMALL OBLIQUE PENCILS—CAUSTICS.

57. WHEN a pencil of rays (Art. 13) is reflected or refracted at the surface of a medium, the reflected or refracted rays will not, *in general,* pass accurately through one point. This peculiarity is sometimes called *astigmatism.* The laws of reflexion and refraction would enable us to determine the direction of any particular ray after its reflexion or refraction, and, as in the case of any system of lines generated according to a determinate law, the consecutive rays after reflexion or refraction will each touch some surface as their *envelope:* this envelope being in fact the locus of their consecutive ultimate intersections. (Todhunter's *Diff. Calc.* Chap. XXV.)

It will be sufficient for the purpose of illustration to confine our attention here to surfaces of revolution, the incident rays being either parallel to the axis of the surface or diverging from some point in that axis. In such a case, the caustic surface will be one of revolution about the same axis as the surface of reflexion or refraction,—and a section of it by a plane passing through the axis will give the *caustic curve,* which curve is touched by each of the reflected or refracted rays which pass in that plane section.

58. The following would be the general process of determining the caustic surface of a pencil for a given reflecting or refracting surface of revolution. In any section through the axis of the pencil, obtain the equation to any reflected or refracted ray referred to axes in that plane; this equation involves the co-ordinates of the point of incidence, which are connected by the equation to the reflecting or refracting curve. Thus the equation to the reflected or refracted ray involves only *one independent parameter,* and the locus of

their ultimate intersections may be obtained in the same way as a common envelope. This envelope is the *caustic curve*, by the revolution of which about the axis the *caustic surface* is generated.

Caustics formed by reflexion were formerly called *catacaustics*; those formed by refraction, *diacaustics*.

59. Since the surfaces with which we have to deal are practically limited in extent, it will be convenient to give a few definitions before we proceed to examine the form of oblique pencils, and we shall suppose the surfaces to be either *plane* or *spherical*.

Def. *Oblique* incidence on a reflecting or refracting spherical surface is either *centrical* or *excentrical*. A pencil is incident *centrically* when the axis of the pencil is incident at a definite point of the surface, called *the centre of the face;* in other cases it is incident *excentrically*.

A distinction must be preserved between the *centre of the surface* and the *centre of the face*; the former is the centre of the sphere of which the reflecting or refracting surface is a portion; the latter is a point on the surface itself, generally the point with respect to which the surface is symmetrical.

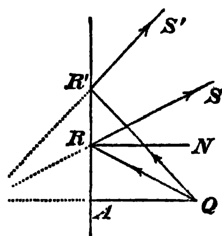
Def. The diameter of the spherical surface which passes through the centre of the face, is in general called the *axis* of the reflecting or refracting surface.

The circumstance which renders the calculation of centrical pencils less complex than those of excentrical pencils is, that in the former case the point of incidence is a definite point of reference from which lines may be conveniently measured, but in the latter it is not so.

60. *If a divergent pencil be incident obliquely on a plane reflecting surface, it will diverge from a point after reflexion.*

Let Q be the origin of a pencil whose axis QA is incident directly on a plane reflecting surface AR'. Then after reflexion the pencil will diverge from a point q in QA

produced, at a distance $Aq = AQ$ (Art. 16). If now we suppose the whole pencil to be removed with the exception of the oblique pencil QRR', the course of this portion of the pencil will remain unaltered, or the oblique pencil will after reflexion diverge from q. Similarly if an oblique pencil *converging* to q be reflected at the plane surface, it will after reflexion converge to Q.

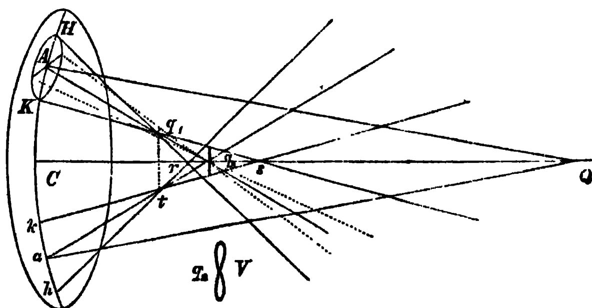


COR. Since $\angle RQR' = \angle RqR'$, the degree of divergence of the reflected pencil is the same as that of the incident pencil.

61. Other cases of oblique reflexion or refraction are not so simple as that of the preceding Article; we shall now shew that if the pencil be *small*, it will after reflexion or refraction *converge to* or *diverge from* two very small straight lines, perpendicular to one another and in different planes, so that the direction of every ray passes through each of these straight lines.

62. *To explain the formation of focal lines, when a small oblique pencil is reflected at a spherical surface, or refracted at a plane or spherical surface.*

Let QC be the axis of a pencil incident directly at C on a spherical reflecting surface, *or on a plane or spherical refracting surface.* If this pencil be supposed to consist of a series of conical surfaces of rays with QC for their common



axis, since all the rays in any such surface will be reflected *or refracted* similarly about QC, the directions of the reflected *or refracted* rays will form a series of conical surfaces Hrh, Aq_2a, Ksk... having a common axis QC along which their vertices r, q_2, s... are arranged. The consecutive intersections of these successive conical surfaces form the caustic surface (Art. 57).

(i) If instead of the whole pencil we consider that portion of it only which is incident on the annulus of the reflecting *or refracting* surface, which would be generated by the revolution of HK about QC, we have corresponding to this an annulus of the caustic surface through some point of which the direction of each ray of the conical shell of reflected *or refracted* rays now considered passes. If HK be *small*, this annulus of the caustic surface may be regarded as a circle in a plane perpendicular to QC, and whose diameter is represented in the figure by q_1t.

(ii) Instead of the conical shell of light incident on the above annulus, let us now consider a small portion thereof incident about HK, by which we come to the case of *a small oblique pencil* whose axis after reflexion *or refraction* is Aq_1q_2.

The direction of every reflected *or refracted* ray now considered will pass through some point of a small circular arc at q_1, which may approximately be regarded as a straight line perpendicular to the plane QCA. This line is called the *Primary Focal Line*, and the point q_1 *the Primary Focus*.

Again, a section of the pencil by a plane through q_2 parallel to the tangent plane of the surface at A, though actually a very elongated *figure of eight*—as indicated at V in the figure,—may very approximately be regarded as a straight line, and is called *the Secondary Focal Line*,—and the point q_2, where the axis of the reflected *or refracted* pencil cuts QC, is called *the Secondary Focus*.

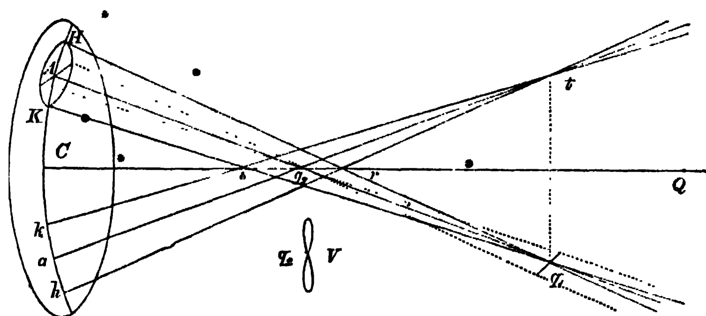
A section of the reflected *or refracted* pencil taken through the axis QC of the reflecting *or refracting* surface, would be strictly a straight line rs,—but it is convenient for our calculations to consider sections parallel to the reflecting *or refracting* element HK—and to treat the section through q_2 as the *Secondary Focal Line*.

Def. With respect to the small oblique pencil of which Aq_1q_2 is the axis,—the plane QCA which contains the axis of the incident pencil and also the axis of the reflected *or refracted* pencil, as well as QC the axis of the spherical surface, is called the *Primary Plane*:—and a plane passing through the axis Aq_1q_2 at right angles to this primary plane is called the *Secondary Plane.*

Hence a small oblique pencil after reflexion or refraction converges to or diverges from two straight lines called *focal lines*;—The *Primary* and *Secondary* focal lines being at right angles to the *Primary* and *Secondary* planes respectively.

63. *Remark.* When the aberration of a direct pencil is *towards* the surface, the primary focus of a small oblique pencil from the same origin is nearer to the surface than the secondary focus, and *vice versa.*

The annexed figure will illustrate the reasoning of the preceding Article, when the aberration of a direct pencil is *from* the surface.



The dotted lines in each figure are intended to represent the bounding rays of a section of the oblique pencil made by a plane through its axis Aq_1q_2 and perpendicular to the Primary Plane,—i.e. by the *Secondary Plane.*

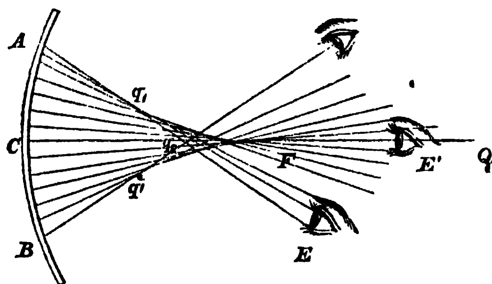
64. *Circle of Least Confusion.*

If a section be taken of the reflected or refracted pencil (Art. 62) by a plane parallel to the tangent plane to the reflecting or refracting surface at A (Figs. Art. 62, 63),—when the plane is drawn through q_1, the section, as we have seen, is

approximately a straight line perpendicular to the primary plane. If this plane be supposed to move gradually from q_1 to q_2,—remaining parallel to itself,—the breadth of the section increases in the primary plane, and decreases in the perpendicular direction until at q_2 the section becomes a straight line in the primary plane. At some point therefore in q_1q_2, the breadth of the section *in* and *perpendicular to* the primary plane is the same, and the section very nearly circular. This section of the pencil is called the *Circle of Least Confusion.*

Obs. The caustic curve is the locus of the primary focus of small oblique pencils reflected or refracted at consecutive small elements of the surface. This suggests another method of finding the equation to the caustic, which is sometimes convenient.

65. We can now explain the nature of the pencils by which an eye sees the image of an object by reflexion *or refraction.*



Let the figure represent a plane section through the axis of a pencil of rays diverging from a point Q, and reflected at the spherical surface ACB (*or refracted at a plane or spherical refracting surface*). The successive reflected rays will be successive tangents to the caustic curve Fq_1, Fq', the cusp of which is at the geometrical focus F.

An eye on the axis at E' will receive a small pencil diverging apparently from F, the geometrical focus, and the image of the point Q seen by E' will coincide with the *geometrical image of* Q (Art. 26). But the case is different with an eye not situated on the axis QC; for instance, an eye E

receives a pencil of rays which is really a small oblique pencil reflected at some part A of the surface, the axis of this small *visual* pencil Eq_2q_1 being a tangent to the caustic curve in the plane passing through the centre of the *pupil* of the eye and the *axis* QC of the surface. This small oblique pencil diverges not from *a point*, but from two focal lines at q_1, q_2, these lines being at right angles to each other and not in the same plane. Hence the image of Q seen by E is more or less indistinct, and is seen in the direction of a tangent drawn from the eye to the caustic of Q in the plane EQC. Since the *circle of least confusion* is the nearest approach to a *point* which such a small oblique pencil admits of, we may perhaps regard the circle of least confusion as the image of Q, —and when an object of *finite* size is viewed by reflexion *or refraction*, we may consider the visible image to be the locus of the circles of least confusion of small oblique pencils emanating from consecutive points of the object; the position of any one of these being determined as above. These circles of confusion will overlap each other, and the image consequently will be more or less confused and indistinct.

We may regard the comparative size of the circles of least confusion in different cases as a measure of the comparative indistinctness of the visible image.

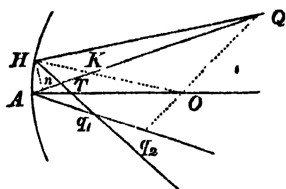
It will sometimes happen that from an eye in a given position more than one tangent can be drawn to the caustic curve in the plane QCE; in such cases there will be more than one visible image.

In the valuable optical diagrams published by *Engel and Schellbach*—(Halle: *first series*, 1850, *second series*, 1856)—the locus of the *primary focus* is taken as the visible image; this can only be received as an *approximate* position of the image; but nevertheless the student will derive great advantage from consulting them. A simplified and less expensive edition of the first series has been published by the *Rev. W. B. Hopkins, late Tutor of St Catharine's College.*

It will be a matter of convenience frequently for us to treat the *primary focus* as the point from which the visual pencil diverges, and regard the visual image as the locus of the primary foci of the small oblique pencils received by the eye from consecutive points of the object.

Determination of the Position of the Foci of Small Oblique Pencils.

66. *A small oblique pencil is reflected at a spherical surface; to find the distances of the foci from the point of incidence of the axis.*



Let Q be the origin of a small pencil whose axis QA is incident at A obliquely on a spherical reflecting surface whose centre is O.

Let Aq_2 be the direction of the axis after reflexion, cutting QO produced in q_2, *the secondary focus* (Art. 62): QH another ray incident in the primary plane and reflected in the direction Hq_1 which cuts Aq_2 in q_1, the primary focus. Join OA, OH, and draw Hn perpendicular to AQ. Let QA, HO intersect in K, and Hq_1, AO in T.

Let $AQ = u$, $Aq_1 = v_1$, $Aq_2 = v_2$, $AO = r$, $AOq_2 = \theta$,

$\left.\begin{matrix}\phi \\ \phi + \delta\phi\end{matrix}\right\}$ the angles of incidence *or reflexion* of $\left\{\begin{matrix}QA \\ QH\end{matrix}\right.$.

Then $\angle HQA = \dfrac{Hn}{HQ}$ ultimately,

$$= \frac{AH.\sin HAQ}{HQ} \text{ ultimately} = \frac{AH.\cos\phi}{u} \text{ ultimately;}$$

$$\therefore \delta\phi = QHO - QAO = \angle K - AQH - (\angle K - AOH)$$

$$= AOH - AQH = \frac{AH}{r} - \frac{AH.\cos\phi}{u}.$$

Again, $\delta\phi = q_1HO - q_1AO = \angle T - AOH - (\angle T - Aq_1H)$

$$= Aq_1H - AOH = \frac{AH.\cos\phi}{v} - \frac{AH}{r}.$$

Equating these values of $\delta\phi$, we get

$$\frac{\cos\phi}{} - \frac{1}{} = \frac{1}{} - \frac{\cos\phi}{}.$$

$$\therefore \frac{1}{v_2} + \frac{1}{u} = \frac{2}{r\cos\phi} \quad \text{..............(i).}$$

Further,
$$\frac{r}{v_2} = \frac{AO}{Aq_2} = \frac{\sin(\theta+\phi)}{\sin\theta};$$

$$\frac{r}{u} = \frac{AO}{AQ} = \frac{\sin(\theta-\phi)}{\sin\theta};$$

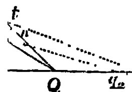
$$\therefore \frac{r}{v_2} + \frac{r}{u} = \frac{\sin(\theta+\phi)+\sin(\theta-\phi)}{\sin\theta} = 2\cos\phi;$$

$$\therefore \frac{1}{v_2} + \frac{1}{u} = \frac{2\cos\phi}{r} \quad \text{.................(ii).}$$

The equations (i), (ii), give respectively the distances of the *primary* and *secondary* foci from A.

67. *A small pencil is incident obliquely on a plane refracting surface; to find the distances of the focal lines from the point of incidence of the axis.*

Let Q be the origin of a small pencil whose axis QA is incident obliquely at A on a plane refracting surface. Draw QC perpendicular to the surface, and produce it backward to cut the direction of QA after refraction in q_2, *the secondary focus.*



Let QH be any ray of the pencil in the primary plane whose direction after refraction cuts Aq_2 produced in q_1, *the primary focus.* Draw An perpendicular to QH, and At perpendicular to Hq_1.

Let $AQ = u$, $Aq_1 = v_1$, $Aq_2 = v_2$,

$\left.\begin{matrix}\phi \\ \phi+\delta\phi\end{matrix}\right\}$ the angles of incidence of $\left\{\begin{matrix}QA \\ QH\end{matrix}\right.$,

$\left.\begin{matrix}\phi' \\ \phi'+\delta\phi'\end{matrix}\right\}$ .... refraction ....

$$\text{Now } \delta\phi = HQA = \frac{An}{AQ} \text{ approximately} = \frac{AH\cos\phi}{u},$$

$$\delta\phi' = Hq_1A = \frac{At}{Aq_1} = \frac{AH\cos\phi'}{v_1}.$$

If μ be the index of refraction from the first medium into the second,

$$\sin\phi = \mu\sin\phi' \text{ (Art. 9)},$$

and by differentiating this we get the relation between the small corresponding variations $\delta\phi$, $\delta\phi'$, viz.

$$\cos\phi\,.\,\delta\phi = \mu\cos\phi'\,.\,\delta\phi';$$

therefore substituting the above values of $\delta\phi$, $\delta\phi'$,

$$\frac{\mu\cos^2\phi'}{v_1} - \frac{\cos^2\phi}{u} = 0 \text{ ....(i).}$$

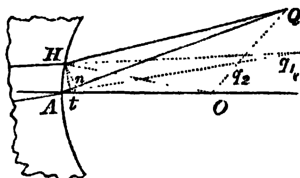
Again

$$\frac{u}{v_2} = \frac{AQ}{Aq_2} = \frac{\sin\phi'}{\sin\phi} = \frac{1}{\mu};$$

$$\therefore \frac{\mu}{v_2} - \frac{1}{u} = 0 \text{ ....(ii).}$$

The results (i), (ii) give respectively the distances of the *primary* and *secondary* foci from A.

68. *A small pencil is obliquely refracted at a spherical surface; to find the distances of the foci from the point of incidence of the axis.*



Let Q be the origin of a small pencil whose axis QA is incident obliquely at A on a spherical refracting surface whose centre is O. Let QO cut the direction of the axis QA after refraction in q_2, *the secondary focus.* Let QH be any ray in the primary plane whose direction after refraction cuts Aq_2 in q_1, *the primary focus.* Join AO, HO, and draw

Hn perpendicular to AQ and Ht perpendicular to Aq_1. Let OH, AQ intersect in K, and OH, Aq_1 in T.

Let $AQ = u$, $Aq_1 = v_1$, $Aq_2 = v_2$, $AO = r$, $AOq_2 = \theta$,

$$\left.\begin{matrix}\phi \\ \phi + \delta\phi\end{matrix}\right\} \text{ angles of incidence of } \left\{\begin{matrix}QA \\ QH\end{matrix}\right.,$$

$$\left.\begin{matrix}\phi' \\ \phi' + \delta\phi'\end{matrix}\right\} \text{ .... refraction.........}$$

Then
$$\angle HQA = \frac{Hn}{HQ} \text{ approximately,}$$

$$= \frac{AH \cos\phi}{u} \text{ ultimately;}$$

$$Hq_1A = \frac{AH \cos\phi'}{v_1}.$$

But $\delta\phi = QHO - QAO = \angle K - HQA - (\angle K - AOH)$

$$= AOH - HQA = \frac{AH}{r} - \frac{AH\cos\phi}{u},$$

and $\delta\phi' = q_1HO - q_1AO = \angle T - Hq_1A - (\angle T - AOH)$

$$= AOH - Hq_1A = \frac{AH}{r} - \frac{AH\cos\phi'}{v_1}.$$

And from the relation $\sin\phi = \mu \sin\phi'$, we get by differentiation

$$\cos\phi \,.\, \delta\phi = \mu \cos\phi' \,.\, \delta\phi'.$$

Substituting the values of $\delta\phi$, $\delta\phi'$, found above, we have

$$\left(\frac{1}{r} - \frac{\cos\phi}{u}\right)\cos\phi = \left(\frac{1}{r} - \frac{\cos\phi'}{v_1}\right)\mu\cos\phi',$$

or
$$\frac{\mu\cos^2\phi'}{v_1} - \frac{\cos^2\phi}{u} = \frac{\mu\cos\phi' - \cos\phi}{r} \qquad \text{...........(i).}$$

Again, $\frac{r}{v_2} = \frac{AO}{Aq_2} = \frac{\sin(\phi' + \theta)}{\sin\theta} = \cos\phi' + \sin\phi' \cot\theta,$

$$\frac{r}{u} = \frac{AO}{AQ} = \frac{\sin(\phi + \theta)}{\sin\theta} = \cos\phi + \sin\phi\cot\theta;$$

therefore remembering that $\sin\phi = \mu \sin\phi'$, we have, by eliminating $\cot\theta$,

$$\frac{\mu}{v_2} - \frac{1}{u} = \frac{\mu\cos\phi' - \cos\phi}{r} \quad \text{............(ii).}$$

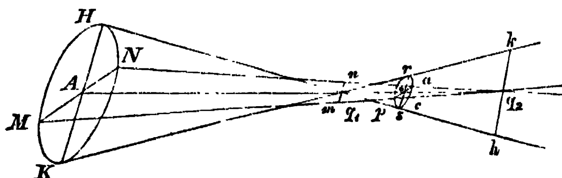
The results (i), (ii) give respectively the distances of the *primary* and *secondary* foci from A.

Note. We may easily obtain (ii) the equation for determining v_2 from the relation $\Delta AOQ = \Delta AOq_2 + \Delta q_2 AQ$.

69. The figures in the preceding two Articles have been drawn to suit the case of refraction from a rarer into a denser medium, in which case $\mu > 1$. The results are equally true for refraction from a denser into a rarer medium. We recommend it as an exercise to the student to go through the investigations for the several cases that can occur, when the incident pencil is *divergent* or *convergent*, the surface *concave* or *convex*, and $\mu > 1$ or $\mu < 1$.

70. *To calculate the position and dimensions of the circle of least confusion of a small oblique pencil reflected at a spherical surface or refracted at a plane or spherical surface.*

Let A be the point of incidence of the axis of a small oblique pencil on a spherical reflecting surface or a plane or



spherical refracting surface. $KMHN$ the section of the incident pencil made by the surface which will approximately be an ellipse with its major and minor axes HK, MN, in and perpendicular to the primary plane.

Suppose mq_1n, hq_2k the primary and secondary focal lines, and ros, poq the breadths in and perpendicular to the primary plane of a section of the pencil through a point o of its axis by a plane parallel to the tangent plane to the surface at A.

Let $$Aq_1 = v_1,\ Aq_2 = v_2,\ MN = \lambda,$$

ϕ = angle of incidence of the axis of the pencil.

Now if the incident pencil be small and its origin distant, it may be considered approximately a right circular cylinder, for the purpose of comparing the dimensions of the section $KMHN$: and this then being a section of the pencil by a plane inclined at an angle $\frac{\pi}{2} - \phi$ to its axis, we have

$$HK = MN \sec\phi = \lambda \sec\phi.$$

By similar triangles

$$\frac{pq}{\lambda} = \frac{oq_2}{v_2},\quad \frac{rs}{\lambda\sec\phi} = \frac{oq_1}{v_1}.$$

If $prqs$ be the circle of least confusion, i.e. $pq = rs$;

$$\therefore \frac{oq_2}{oq_1} \cdot \frac{v_1}{v_2} = \sec\phi = \frac{v_1}{v_2} \cdot \frac{v_2 - Ao}{Ao - v_1},$$

whence $$Ao = \frac{v_1 v_2 (1 + \cos\phi)}{v_1 \cos\phi + v_2} \quad\text{.................. (i)},$$

and $$pq = \frac{\lambda}{v_2}(v_2 - Ao) = \lambda \frac{v_2 - v_1}{v_1\cos\phi + v_2} \quad\text{......... (ii)}.$$

(i), (ii) give respectively the distance from A of the circle of least confusion and the diameter of this circle.

Cor. 1. If $\alpha_1 = mn$, $\alpha_2 = hk$, the *lengths* of the primary and secondary focal lines, we have by similar triangles

$$\frac{\alpha_1}{\lambda} = \frac{v_2 - v_1}{v_2}, \text{ and } \frac{\alpha_2}{\lambda\sec\phi} = \frac{v_2 - v_1}{v_1},$$

which give α_1, α_2.

COR. 2. If the pencil be cylindrical at incidence, and be reflected at a spherical surface, $u = \infty$, and

$$\frac{1}{v_1} = \frac{2}{r \cos \phi}, \quad \frac{1}{v_2} = \frac{2 \cos \phi}{r} \text{ (Art. 66)};$$

$$\therefore Ao = \frac{r}{2} \cdot \frac{(1 + \cos \phi) \cos \phi}{1 + \cos^2 \phi}, \quad pq = \lambda \frac{\sin^2 \phi}{1 + \cos^2 \phi},$$

$$\alpha_1 = \lambda \sin^2 \phi, \quad \alpha_2 = \lambda \cdot \frac{\sin^2 \phi}{\cos^2 \phi}.$$

COR. 3. Since $\dfrac{v_2 - Ao}{Ao - v_1} = \dfrac{v_2 \sec \phi}{v_1}$, and this latter quantity approaches unity as ϕ is diminished,

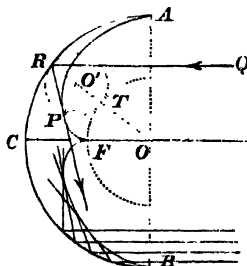
$$\therefore \text{ ultimately } AO = \tfrac{1}{2}(v_1 + v_2),$$

i.e. the centre of the circle of least confusion has the middle point between the two foci for its limiting position, and it may approximately be supposed to have this position when the obliquity (ϕ) of the pencil is small.

71. *Examples and Illustrations.*

Parallel rays are incident on a reflecting semicircular mirror, and in its plane; to find the caustic curve.

Let QR be any ray of the beam of parallel rays incident at R on the mirror ACB, whose centre is O; CO the diameter of the mirror parallel to the incident rays.



Describe a circle with centre O and radius $OF = \frac{1}{2} . CO$.

Join RO, and upon RT as diameter describe a circle, O' its centre; therefore $O'T = \frac{1}{2} . TO$.

Let RP be the direction of the ray QR after reflexion. Join $O'P$, PT.

Now $\angle PO'T = 2PRT = 2QRO = 2TOF$,

and $OT = 2O'T$; $\therefore$ arc TF = arc TP.

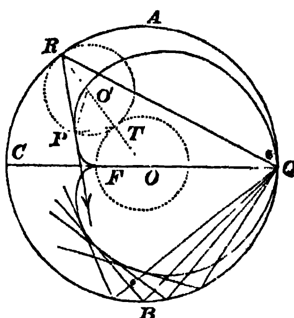
We may then suppose the circle RPT to roll upon the circle TF, the point P coinciding initially with F, the geometrical focus of rays incident at C, so that P traces out the epicycloid FPA, FB,—the cusp of which is at F. And since RPT is a right angle and PT is the direction of the normal to the path of P, therefore RP is a tangent to the path of P, i.e. each reflected ray *touches* the epicycloid $APFB$, which is therefore the caustic curve.

This caustic has a cusp at F, and touches the circular mirror at A and B. By the revolution of this curve about CO we should get the caustic surface of parallel rays reflected at a hemispherical mirror.

Obs. The caustics formed by reflexion may be easily shewn experimentally by taking a narrow strip of polished steel —(a piece of watch-spring for instance)—bent into any concave form. Place it upright on a sheet of paper, and let it be exposed to the rays of the sun, so that the plane of the paper passes nearly but not quite through the sun; the caustic will be seen traced on the paper and marked by a bright, well-defined line; the part within being brighter than that without, and the light diminishing by rapid gradations from the caustic inwards. If the form of the spring be varied, varieties of catacaustics with their singular points, cusps, contrary flexures, &c. will be seen beautifully developed. The bright line seen on the surface of a drinking-glass nearly full of liquid, standing in the sunshine, is a familiar instance of the caustic of a circle.

72. *To find the form of the caustic, when the reflecting curve is a circular arc, and the rays diverge from a point in its circumference.*

Let QR be any ray diverging from Q, a point in the circumference of the circular mirror ACB whose centre is O. Join OR, and with centre O and radius $= OF = \frac{1}{3} . OC$ describe a circle; describe another circle on RT as diameter, so that its radius $O'T = TO$.



RP the direction of QR after reflexion, join $O'P$, PT.

Then

$$\angle\, PO'T = 2PRT = 2QRO = TOF,$$

and $$O'T = OT;$$

$$\therefore \text{arc } TP = \text{arc } TF.$$

Hence it will readily appear that the reflected rays all touch the epicycloid traced out by the point P of the circle RPT, as this circle rolls upon FT fixed. This epicycloid is the caustic curve required; it has a cusp at F, the geometrical focus of rays reflected at C, and touches the mirror at Q.

We can easily find the polar equation to this caustic.

Suppose P, F joined,

and let $$PF = r,\ \angle\, PFC = \theta,\ OQ = a,$$

since $$OF = OT = O'T = O'P = \frac{a}{3},$$

and $$\angle\, FOT = 2ORQ = 2PRO' = \angle\, PO'T,$$

$$PF \text{ is parallel to } OO', \text{ and } \theta = \angle\, PFC = \angle\, FOT;$$

$$\therefore \frac{2}{3}a = OO' = PF + 2FO\cos FOT = r + \frac{2}{3}a\cos\theta;$$

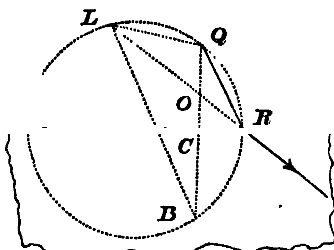
$$\therefore r = \frac{2}{3}a\,(1 - \cos\theta),$$

the equation to the caustic, which is a cardioid.

73. *Rays diverging from a point are refracted at a plane surface, the caustic is the evolute of a conic section.*

(i) Suppose the refraction to take place from a denser into a rarer medium. Let QR be any ray diverging from Q and incident at R, on the plane surface. Draw QCB perpendicular to the surface, and make $CB = QC$. Describe

a circle about QRB, and let LR be the direction of the refracted ray.



Join QL, LB; O the point of intersection of QC, LR.

Since LR bisects $\angle QLB$,

we have $\frac{OB}{OQ} = \frac{LB}{LQ}$;

$$\therefore \frac{QB}{OQ} = \frac{BL + QL}{QL};$$

$$\therefore \frac{QB}{BL + QL} = \frac{OQ}{QL} = \frac{\sin QLR}{\sin QOL}.$$

And $QLR = QBR = RQB = \phi$, the angle of incidence of QR

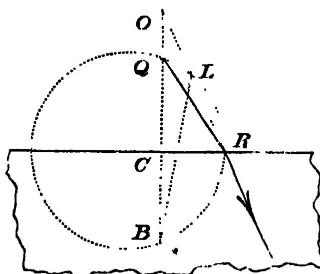
$$QOL = \phi' = \angle \text{ of refraction of } QR;$$

$$\therefore \frac{QB}{BL + QL} = \frac{\sin \phi}{\sin \phi'} = \mu, \text{ (where } \mu < 1).$$

Hence $BL + QL = \frac{BQ}{\mu}$ = a constant quantity, and the locus of L is an ellipse of which Q, B are the foci,—and LR is a normal at L to the locus of L, since it bisects the angle between the focal distances; that is, the caustic curve is the evolute of the ellipse whose foci are Q, B and axis major

$$= \frac{BQ}{\mu}, \text{ and eccentricity} = \mu.$$

(ii) If the refraction takes place from a rarer into a denser medium, let LR be the direction of the refracted ray, produce LR to meet CQ produced in O,—then it may be shewn that LO bisects the exterior angle QLB, and



$$\frac{BL}{QL} = \frac{BO}{QO};$$

$$\therefore \frac{BL-QL}{QL}=\frac{BQ}{QO};$$

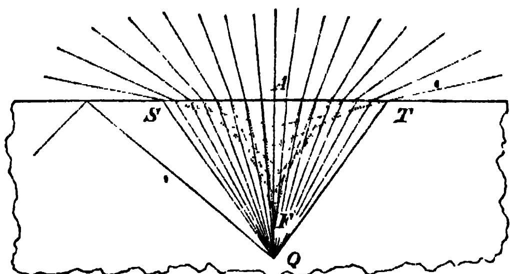
$$\therefore \frac{QB}{BL-QL}=\frac{QO}{QL}=\frac{\sin OLQ}{\sin LOQ}=\frac{\sin \phi}{\sin \phi'}=\mu, \ (\mu>1).$$

Hence the locus of L is a hyperbola, and OL is the direction of the normal at L; that is, the caustic curve is the evolute of a hyperbola whose foci are Q, B, and axis major

$$=BL-QL=\frac{QB}{\mu}, \text{ and eccentricity}=\mu.$$

74. *Illustration.*

Thus, if Q be a radiant point, the rays from which are refracted at a plane surface ST into a rarer medium, the emergent rays will be tangents to a *virtual* caustic SF, FT, which is a portion of the evolute of an ellipse, one cusp of

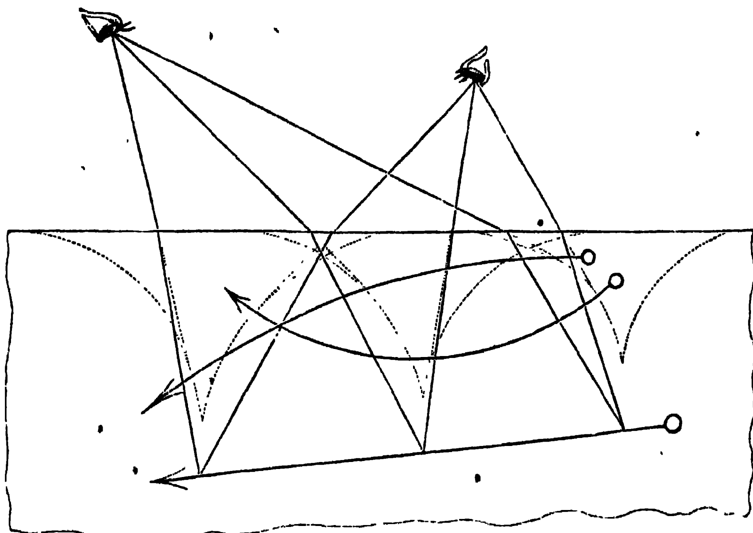


which is at F the geometrical focus of Q. The branches of the caustic touch the plane surface of the medium at S, T, points determined by the condition that the rays QS, QT are incident at the critical angle. (Art. 80.)

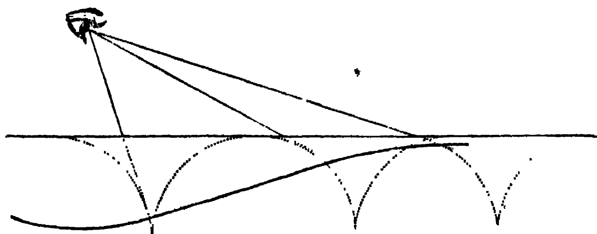
If we suppose the figure to revolve about QFA the axis of the surface, the curve SFT will trace out the caustic surface, to which every ray of the beam of refracted rays is a tangent.

Rays incident beyond S, ' are internally reflected within the denser medium.

The caustic just obtained enables us to illustrate the deformation of the visible image of an object situated in a denser medium than that in which the eye is situated, the media being bounded by a plane. The dotted curves in the figure are intended to represent the caustics of three different points of the object,—and to avoid confusion, only the *axes* of the pencils which the eye receives are drawn.



Again, suppose a surface of still water with a level horizontal bottom not very deep; the bottom will not appear a



plane, but will seem to rise on all sides, being shaped something like a basin below the eye, and tending at distant points towards the surface of the water as an asymptotic plane.

75. *If a small pencil of diverging rays be reflected at a concave spherical mirror, to find the limits of the distance of the origin from the point of incidence in order that the reflected pencil may converge to, or diverge from,* both *the focal lines.*

The distances of the primary and secondary foci of the reflected pencil from the point of incidence of its axis are given by the equations

$$\frac{1}{v_1} = \frac{2}{r\cos\phi} - \frac{1}{u}, \quad \frac{1}{v_2} = \frac{2\cos\phi}{r} - \frac{1}{u}.$$

Now the reflected pencil *converges* to each focal line, if v_1 and v_2 are both positive; this will be the case if u be $> \frac{r}{2\cos\phi}$, and consequently $> \frac{r\cos\phi}{2}$,... and it *diverges* from each line if v_1 and v_2 are both negative, which will be the case if u be $< \frac{r\cos\phi}{2}$, and consequently $< \frac{r}{2\cos\phi}$. If u be $> \frac{r\cos\phi}{2}$ and $< \frac{r}{2\cos\phi}$, the reflected pencil converges to one of the focal lines and diverges from the other.

If the rays reflected in the primary plane are parallel or v_1 be infinite, then $u = \frac{r\cos\phi}{2}$ and $v_2 = -\frac{r\cos\phi}{2\sin^2\phi}$; the latter is the polar equation to the locus of the secondary focus, when the origin assumes different positions in the same primary plane, so that the rays in the primary plane may be parallel.

See a paper on the *Cyclide* by J. C. Maxwell, *Quarterly Journal of Mathematics*, Vol. IX. p. 111.

CHAPTER V.

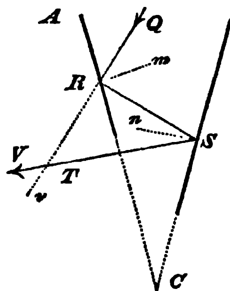
SUCCESSIVE REFLEXIONS AND REFRACTIONS;—PRISMS—LENSES.

76. IN this section we proceed to examine the modification in direction and form which a pencil undergoes, after being reflected or refracted more than once at plane or spherical surfaces.

Def. When a ray has had its direction altered by reflexion or refraction, its *deviation* is the angle between its present direction and its original direction produced.

77. *Successive Reflexions at Plane Surfaces.*

If a pencil be reflected once by each of two plane surfaces to find the deviation of its axis: supposing its course to be in one plane perpendicular to the intersection of the surfaces.



Let $QRSTV$ be the course of the axis of a pencil reflected at R and S at two plane surfaces in a plane which cuts these surfaces perpendicular in CA, CB. Then the angle VTv, measured from RTv, the direction of QR produced, towards the point V,—is the deviation of the axis of the pencil after the two reflexions.

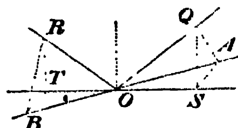
Draw Rm, Sn normals to the reflecting planes at R, S.

$$\text{Now, } \angle\, vTV = \angle\, QRS - \angle\, RST$$
$$= 2\,\angle\, SRm - 2\,\angle\, RSn$$
$$= 2\left(\frac{\pi}{2} - SRC\right) - 2\left(\frac{\pi}{2} - RSB\right)$$
$$= 2\,(RSB - SRC) = 2\,\angle\, ACB;$$

i.e. the deviation of the axis of the pencil is double the inclination of the reflecting planes.

COR. The degree of divergence of the pencils is unaltered by the reflexions. (Art. 60. Cor.)

78. *When a ray is reflected at a plane surface, the incident and reflected ray are equally inclined to any the same line which is parallel to the reflecting plane.*



Let the plane which passes through QO, OR, the incident and reflected rays, meet the reflecting plane to which it is perpendicular in the line SOT.

Through the point O draw the line AOB in the reflecting plane, parallel to the given line.

Take $OR = OQ$ and through R, Q draw planes at right angles to the line SOT, and meeting AOB in B, A. Then remarking that each of the angles at S, T is a right angle, and that $OR = OQ$ it will easily be seen that the triangles ROT, QOS are similar and equal, as are also BOT and AOS, and consequently RTB, QSA are so likewise; whence

$$\angle\, ROB = \angle\, QOA,$$

i.e. QO, OR are equally inclined to AB, and the same is true with respect to any line parallel to AB.

Hence when a ray is reflected at two plane surfaces in succession, the inclination of the ray to the line of intersection of the surfaces before the first and after the second reflexion is the same; and a similar result may be inferred after

reflexion at any number of plane surfaces in succession, if the lines of intersection of the surfaces be all parallel.

A familiar instance of rays so reflected is afforded by the *Kaleidoscope.*

Successive Refractions at Plane Surfaces.

79. In examining the effect of successive refractions on a pencil, the following two considerations are employed:—

(i) The geometrical focus of a direct pencil being the point of ultimate intersection of any refracted ray with the axis, the pencil after one refraction may ultimately be considered to diverge from or to converge to this point as an origin, the pencil being supposed a very small one.

(ii) If a ray be reflected or refracted in any manner in passing from one point to another, it is assumed that it might pass by the same course reversed from the latter point to the former. This is in accordance with a general law in Optics, that the visibility of two points from one another is mutual; in other words, if a ray of light proceeding from A arrives by any course at B however often reflected or refracted, a ray can also arrive at A from B by retracing precisely the same course in the opposite direction.

Hence when a pencil is emerging from a refracting medium, we may when convenient reduce this case to the more familiar one of a pencil entering a refracting medium, by supposing the course of each ray reversed.

Critical Angle.

80. If ϕ be the angle of incidence of a ray of light, and ϕ' its angle of refraction into a denser medium, μ the refractive index between the media, then

$$\sin \phi = \mu \sin \phi', \text{ or, } \sin \phi' = \frac{1}{\mu} . \sin \phi.$$

Now μ being > 1, (Art. 9), $\sin \phi'$ is < 1, and this equation gives a real angle of refraction for any given angle of inci-

dence; and accordingly refraction into a denser medium is always possible whatever be the angle of incidence.

If the refraction be from a denser into a rarer medium, and if ϕ' be the angle of incidence, ϕ the angle of refraction,

$$\sin \phi = \mu \sin \phi'.$$

This equation does not give a real angle of refraction for a given angle of incidence ϕ', unless ϕ' be such that

$$\mu \sin \phi' \text{ be not} > 1, \text{ or } \phi' \text{ be not} > \sin^{-1}\frac{1}{\mu}.$$

Accordingly if the angle of incidence in the denser medium exceed this limit, it is found that refraction does not take place, but that the ray is reflected within the denser medium at the surface which separates the media.

Def. The angle $\sin^{-1}\frac{1}{\mu}$, which the angle of incidence in the denser medium must not exceed, in order that refraction into a rarer medium may be possible, is called the *critical angle* of the media between which the refractive index is μ.

The critical angle for water is about 48° 27′ 40″,
..............................crown glass 40° 30′
for chromate of lead it does not exceed 19° 28′ 20″.

81. When light is incident at the surface of a medium at an angle greater than the critical angle the reflexion is total, and is much more brilliant than that obtained in any other way—as at the surface of quicksilver or of any polished metal. It may be exhibited in a simple manner by holding a glass of water above the level of the eye; the under surface of the water will appear very bright from the light internally reflected, and any object in the water—a spoon for instance—will be seen by reflexion at the under surface, more brilliantly than it would by reflexion at any mirror. Or again, if a glass prism be held so that the eye may receive light passing through it after internal reflexion at one of its faces, that face will appear as bright as polished silver.

This property of internal reflexion is employed in the camera lucida, in diagonal eye-pieces for telescopes, &c., with great advantage.

From the preceding we may explain some of the peculiarities which would present themselves to an eye under the surface of still water. All external objects would appear compressed within a conical space whose axis is vertical and vertical angle $= 2\sin^{-1}\frac{1}{\mu} = 97^{\circ}$ nearly, the objects near the horizon being much distorted and contracted, especially in height.

Beyond this conical space, objects within the water would be seen by reflexion at the surface.

82. *Def.* A portion of a refracting medium contained between two parallel plane surfaces is called a *plate*.

It is a result of experiment that when a ray of light passes through any number of media separated by parallel plane surfaces, if any two of these media be identical, the directions of the ray in them are parallel.

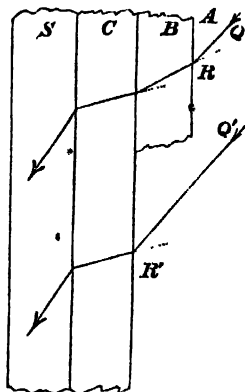
This may be illustrated experimentally by holding a *plate* of glass or any transparent substance before the object-glass of a telescope directed to a distant object or before the naked eye,—the apparent place of the object will be the same, whatever be the inclination to the visual ray at which the plate is held.

83. The experimental result just referred to may be employed to obtain a relation between the indices of refraction of successive media, as follows:—

Let A, B, C, S be four media bounded by parallel planes, and let A and S be identical. Then if QR, $Q'R'$ be two rays parallel in A, their directions in S will be parallel after one of them QR has been refracted through B, C, and the other $Q'R'$ refracted at once into C.

But the rays might follow the same course in a reversed direction, in which case the angles of incidence from S to C being the same, the angles of refraction are the same, and each of the rays when passing though C has undergone the same deviation from its direction in A.

Let ϕ_1, ϕ_2, ϕ_3 be the angles of incidence on B, C, S, and let ${}_\alpha\mu_\beta$ denote the index of refraction from any medium denoted by α into one denoted by β.



Then

$$\frac{\sin\phi_1}{\sin\phi_2} = {}_A\mu_B, \quad \frac{\sin\phi_2}{\sin\phi_3} = {}_B\mu_C;$$

$$\therefore {}_A\mu_C = {}_S\mu_C = \frac{\sin\phi_1}{\sin\phi_3} = {}_A\mu_B \cdot {}_B\mu_C;$$

and a similar result might be obtained, connecting the indices of refraction of any number of media.

Ex. From air to glass $\mu = \frac{3}{2}$, from air to water $\mu = \frac{4}{3}$.

Hence from glass to water,

$${}_G\mu_W = \frac{\text{index from air to water}}{\text{index from air to glass}} = \frac{8}{9}.$$

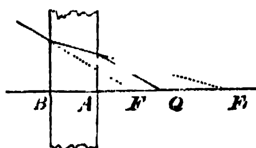
84. It will readily follow from the previous Article that if a ray of light be refracted through any number of media bounded by parallel plane surfaces, its direction in any medium will have undergone the same deviation as if it had been refracted directly into that medium.

Note. In all cases of refraction in which a single medium only is mentioned, the other is supposed to be *vacuum*. In such cases the corresponding value of μ is called the *absolute index* of refraction.

It follows also from the preceding results that the index of refraction from a medium A into another B is the reciprocal of that from B into A—i.e. ${}_A\mu_B \cdot {}_B\mu_A = 1$: a conclusion which follows at once from the law of refraction combined with the remark in Art. (79).

85. *To determine the geometrical focus of a pencil after direct refraction through a plate.*

Let Q be the origin of a pencil whose axis QAB passes directly through a refracting plate. Let F_1, F be the geometrical foci of the pencil after refraction at the first and second surfaces respectively.



Let $AQ = u$, $BF = v$, $AB = t$.

From the first refraction Art. (19)

$$AF_1 = \mu u.$$

Now F_1 being regarded as an origin, a pencil diverging from F_1 after refraction at the second surface, has F for its geometrical focus. Hence if the course of the pencil be supposed reversed, a pencil converging to F would after refraction into the plate have F_1 for its geometrical focus; (Art. 79)

$$\therefore BF_1 = \mu . BF, \text{ or } AF_1 + t = \mu v;$$

$$\therefore t = \mu v - \mu u, \quad \text{or } v = u + \frac{t}{\mu},$$

which determines the position of F.

Note. Since $BF - BQ = -\dfrac{\mu - 1}{\mu} . t$, it follows that an object appears nearer when viewed through a plate denser than the surrounding medium.

86. When a small pencil is refracted obliquely through a portion of medium bounded by two given surfaces, there is an apparent difficulty at the second refraction in consequence of neither the incident nor the refracted pencil having then a *point* of divergence or convergence.

In the primary plane, however, the pencil on account of its smallness may after each refraction be regarded as converging to or diverging from a point, and we may thus apply —with reference to this plane—propositions which suppose

the whole pencil to emanate from a point. Again, the rays incident in a plane perpendicular to the primary plane have, after each refraction, a point of convergence or divergence, and therefore we may use with respect to this plane the results deduced in the previous sections. The *form* of the emergent pencil is thus determined by finding the foci of two sections of it, one by the primary plane, the other by a plane perpendicular to that plane. These foci are separated by an interval which is generally small in the cases which occur in the construction of optical instruments, where the pencils are of small breadth and obliquity.

The results obtained on the above hypothesis are of course only approximate, and become less and less accurate the more numerous the successive refractions to which the pencil is subjected.

87. *To determine the foci of a small pencil refracted obliquely through a plate.*

Let Q be the origin of a small pencil whose axis $QAST$ passes obliquely through a plate, the surfaces of which it cuts at A and S.

Let Q_1, Q_2 be the primary and secondary foci of the pencil after one refraction, q_1, q_2 those of the emergent pencil.

Let $AQ = u$, $Sq_1 = v_1$, $Sq_2 = v_2$, $AB = t$, the thickness of the plate, ϕ, ϕ' the angles of incidence and refraction at A, and of emergence and incidence at S.

From the first refraction $AQ_1 = \dfrac{\mu \cos^2 \phi'}{\cos^2 \phi} u$. (Art. 67.) Now the pencil diverging in the primary plane from Q_1 diverges after refraction at the second surface of the plate from q_1 in the same plane; hence if we suppose the course of the pencil reversed, a pencil converging to q_1 would after refraction into the plate converge in the primary plane to Q_1.

Hence $$SQ_1 = \frac{\mu \cos^2 \phi'}{\cos^2 \phi} v_1;$$

$$\therefore AS \text{ or } t \sec\phi' = SQ_1 - AQ_1 = \frac{\mu \cos^2\phi'}{\cos^2\phi} u . (v_1 - u),$$

$$\text{or } v_1 = u + t . \frac{\cos^2\phi}{\mu \cos^3\phi'} \text{..................} \text{(i).}$$

Similarly $\quad AQ_2 = \mu u, \; SQ_2 = \mu v_2;$

$$\therefore v_2 = u + \frac{t}{\mu \cos\phi'} \text{.....................} \text{(ii).}$$

The results (i), (ii) determine respectively the primary and secondary foci of the emergent pencil.

Prisms.

88. *Defs.* A *prism* is a portion of a refracting medium bounded by two plane surfaces inclined at a finite angle to one another.

The *edge* of the prism is the line in which these two surfaces meet, or would meet if produced.

The two plane surfaces are called the *faces* of the prism and their inclination to one another is the *refracting angle* of the prism.

A plane perpendicular to each of the faces, and therefore to the edge of the prism, is called a *principal section* of the prism or of the two surfaces.

89. *When a ray of light is refracted out of one medium into another, as the angle of incidence increases, the deviation also increases.*

Let ϕ, ϕ' be the angles of incidence and refraction of the ray, then $\sin\phi = \mu \sin\phi'$,—and supposing $\mu > 1$,

$$\frac{\sin\phi - \sin\phi'}{\sin\phi + \sin\phi'} = \frac{\mu - 1}{\mu + 1},$$

$$\text{or } \tan\frac{\phi - \phi'}{2} = \frac{\mu - 1}{\mu + 1} . \tan\frac{\phi + \phi'}{2}.$$

Now if ϕ and consequently ϕ' be increased, $\tan \frac{\phi + \phi'}{2}$ is increased, since $\frac{\phi + \phi'}{2}$ is $< \frac{\pi}{2}$;

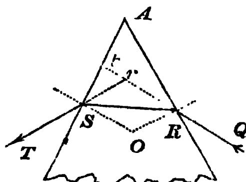
$$\therefore \tan \frac{\phi - \phi'}{2} \text{ is increased } \frac{\mu - 1}{\mu + 1} \text{ being positive};$$

$$\therefore \phi - \phi' \qquad \frac{\phi - \phi'}{2} \text{ being} < \frac{\pi}{2}.$$

i.e.—the deviation is increased if the angle of incidence be increased:—the same result will follow if $\mu < 1$.

90. *The axis of a pencil which passes through a prism in a principal plane, is turned* from *the edge of the prism;—the prism being denser than the surrounding medium.*

Fig. 1.



Let $QRST$ represent the course of the axis of the pencil in a principal plane of the prism. Then the normals at R, S, the points of incidence and emergence, must meet either *within*, *without*, or *upon one face of* the prism.

(i) Let them meet within, as at O; then it is clear that QR being the incident and RS the refracted ray the deviation at R is *from* the edge of the prism; similarly ST being the ray emergent at S, the deviation at S is *from* the edge. Therefore the whole deviation is *from* the edge.

(ii) Let the normals at R, S meet without the prism,

Fig. 2.

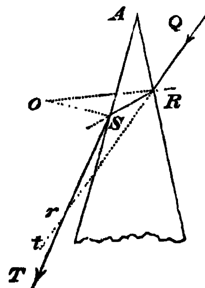
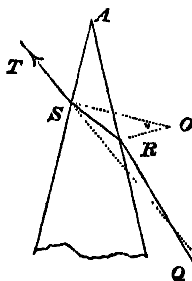


Fig. 3.



as at O,—then it is clear that the angle of refraction at R (fig. 2) is less than that at S; and therefore the deviation at R, which is *towards* the edge, is less (*compare Art.* 89) than the deviation at S, which is *from* the edge,—and *vice versâ* in fig. 3.

Therefore on the whole the deviation is *from* the edge.

(iii) If the normals at R, S meet on one face of the

Fig. 4.

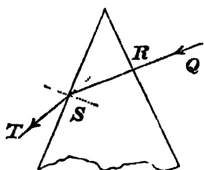
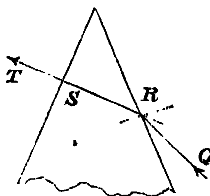


Fig. 5.



prism, then at that point the deviation of the axis of the pencil is *from* the edge, and there is no deviation at the other point, R or S (fig. 4 or 5).

Therefore on the whole the axis is turned from the edge, or towards the thicker part of the prism.

COR. If the prism be rarer than the surrounding medium these effects will be reversed, and the deviation of the axis will be *towards* the edge or *from* the thicker part of the prism.

91. *When a pencil is refracted through a prism in a principal plane, to find the deviation of its axis.*

Let $QRST$ be the course of the axis of the pencil (figs. 1, 2, of previous Article) in a principal plane which meets the faces of the prism in SA, RA. Let QR produced to some point t cut ST, or ST produced backward, in r. Also let the normals to the faces at R and S meet in O.

Let ϕ, ϕ' be the angles of incidence and refraction at R,

ψ, ψ'....................emergence and incidence at S,

$D = \angle trT$ the deviation of the axis,

$i = \angle SAR$ the refracting angle of the prism.

(i) If the normals at R, S intersect within the prism (fig. 1),

$$D = \angle rSR + \angle rRS = \psi - \psi' + \phi - \phi',$$

and since the four angles of a quadrilateral are equal to four right angles;

$$\therefore i = \pi - \angle SOR = \angle OSR + \angle ORS = \phi' + \psi'.$$

(ii) If the normals at R, S intersect without the prism (fig. 2),

$$D = \angle SRQ - \angle rSR = \pi - \phi + \phi' - (\pi - \psi + \psi')$$
$$= \psi - \psi' - (\phi - \phi'),$$

and the inclination of two surfaces being the same as that of their normals,

$$i = \angle ROS = \pi - RSO - SRO = \psi' - \phi'.$$

Therefore in both cases,

$$i = \psi' \pm \phi' \quad \text{.......................................} \quad \text{(i)},$$

and $D = \psi - \psi' \pm (\phi - \phi')$ or $D = \psi \pm \phi - i$ ...... (ii).

These results (i), (ii), combined with $\sin\phi = \mu \sin\phi'$, and $\sin\psi = \mu \sin\psi'$, are sufficient to determine analytically the circumstances of a ray passing through a prism in a principal plane.

92. *Obs.* 1. Since neither ϕ' nor ψ' can exceed a certain magnitude, viz. $\sin^{-1}\frac{1}{\mu}$, it follows that the equation $\phi' + \psi' = i$ will be an impossible one, if i exceed twice the critical angle of the substance of which the prism is composed. When i exceeds this limit the ray cannot pass through the prism in the manner supposed, but will be internally reflected at the second surface.—It may of course emerge after one or more internal reflexions.

Obs. 2. If ϕ' and ψ' be considered positive or negative according as they fall on the side of the normals at R, S

towards or *from* the edge of the prism A, and therefore ϕ and ψ, positive or negative according as they fall on the side of the normals at R and S, *from* or *towards* A, the two cases of the previous investigation may be conveniently included in one formula which will always be algebraically correct, viz.

$$D = \psi + \phi - i, \qquad i = \phi' + \psi',$$

$$\text{and } \sin\phi = \mu \sin\phi', \qquad \sin\psi = \mu\sin\psi'.$$

COR. If ϕ and ψ be *each* small,—in which case i is small,—we have

$$D = \psi + \phi - i = \mu(\psi' + \phi') - i \text{ approximately,}$$

$$= \mu i - i = (\mu - 1)\, i.$$

93. If the deviation be a minimum, since

$$D = \psi + \phi - i, \qquad i = \phi' + \psi',$$

we must have

$$\frac{dD}{d\phi} = 0 = \frac{d\psi}{d\phi} + 1, \quad \therefore \frac{d\psi}{d\phi} = -1 \qquad \ldots.\ \text{(i)},$$

$$0 = \frac{d\phi'}{d\phi} + \frac{d\psi'}{d\phi} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \text{(ii)}.$$

Also

$$\sin\phi = \mu\sin\phi'; \quad \therefore \cos\phi = \mu\cos\phi'\frac{d\phi'}{d\phi} \ldots\ldots\ldots\ \text{(iii)},$$

$$\sin\psi = \mu\sin\psi'; \quad \therefore \cos\psi\frac{d\psi}{d\phi} = \mu\cos\psi'\frac{d\psi'}{d\phi} \ldots\ldots\ \text{(iv)};$$

$$\therefore \frac{d\phi'}{d\phi} = \frac{\cos\phi}{\mu\cos\phi'}, \quad \frac{d\psi'}{d\phi} = -\frac{\cos\psi}{\mu\cos\psi'},$$

whence

$$\frac{\cos\phi}{\mu\cos\phi'} - \frac{\cos\psi}{\mu\cos\psi'} = 0;$$

$$\therefore (1 - \sin^2\psi')(1 - \mu^2\sin^2\phi') = (1 - \sin^2\phi')(1 - \mu^2\sin^2\psi')$$

whence $$\sin^2\phi' = \sin^2\psi',$$

or $$\phi' = \pm\psi'.$$

Since $i = \phi' + \psi'$ the lower sign is inadmissible.

Hence $\phi' = \psi'$,

and therefore $\phi = \psi$ is the only result.

Further, differentiating (i), (ii), (iii), (iv),

$$\frac{d^2D}{d\phi^2} = \frac{d^2\psi}{d\phi^2}, \quad \frac{d^2\phi'}{d\phi^2} + \frac{d^2\psi'}{d\phi^2} = 0;$$

$$\cos\psi \cdot \frac{d^2\psi}{d\phi^2} - \sin\psi \cdot \left(\frac{d\psi}{d\phi}\right)^2 = \mu\cos\psi' \frac{d^2\psi'}{d\phi^2} - \mu\sin\psi'\left(\frac{d\psi'}{d\phi}\right)^2;$$

$$-\sin\phi = \mu\cos\phi' \cdot \frac{d^2\phi'}{d\phi^2} - \mu\sin\phi' \cdot \left(\frac{d\phi'}{d\phi}\right)^2.$$

Putting $\phi = \psi$, $\phi' = \psi'$, we get for the value of $\frac{d^2D}{d\phi^2}$,

$$\frac{d^2D}{d\phi^2} = \frac{2(\mu^2 - 1)\sin\phi}{\mu^2\cos^2\phi'\cos\phi} = \text{a positive quantity.}$$

Hence $\phi = \psi$ makes D a *minimum*.

That is, when the angle of incidence is equal to the angle of emergence, the deviation is less than in any other case.

94. We give another method of obtaining the condition of minimum deviation, which does not require the use of the *Differential Calculus*.

With our usual notation

$$D = \psi + \phi - i, \quad i = \phi' + \psi',$$

$$\sin\phi = \mu\sin\phi', \quad \sin\psi = \mu\sin\psi'.$$

Eliminating ϕ', ψ' we get

$$\sin\phi = \mu\sin\phi' = \mu\sin(i - \psi') = \mu\sin i\cos\psi' - \cos i\sin\psi;$$

$$\therefore (\sin\phi + \cos i\sin\psi)^2 + (\sin\psi\sin i)^2 = \mu^2\sin^2 i,$$

or $$\sin^2\phi + 2\cos i\sin\phi\sin\psi + \sin^2\psi = \mu^2\sin^2 i.$$

$$\text{But } \sin^2\phi + \sin^2\psi = \frac{1-\cos 2\phi}{2} + \frac{1-\cos 2\psi}{2}$$

$$= 1 - \cos(\phi - \psi) \,.\, \cos(\phi+\psi),$$

$$2 \sin\phi \sin\psi = \cos(\phi-\psi) - \cos(\phi+\psi).$$

Whence

$$(\mu^2 - 1)\sin^2 i = \{\cos i + \cos(\phi - \psi)\}\{\cos i - \cos(\phi+\psi)\}.$$

Now observing that i is a given quantity, we see that D will be least when $\phi + \psi$ is least, i.e. when $\cos i - \cos(\phi+\psi)$ is least; this will be the case when $\cos i + \cos(\phi - \psi)$ is greatest, i.e. when $\cos(\phi - \psi)$ is greatest, which it will be when $\phi = \psi$.

Hence D is a minimum when $\phi = \psi$, or the angles of incidence and emergence equal.

95. *Obs.* When the deviation is a minimum

$$\phi = \psi, \quad \phi' = \psi', \quad D = 2\phi - i, \quad i = 2\phi',$$

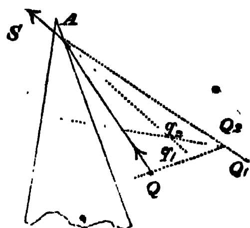
and $$\mu = \frac{\sin\phi}{\sin\phi'} = \frac{\sin \frac{D+i}{}}{\sin}$$

This result affords the most exact means of determining the refractive indices of any substance which can be formed into a prism. The refractive angle of the prism, and the minimum deviation of the ray passing through it in a principal plane must be measured, and thence the value of μ can be calculated. We shall refer to the method of doing this hereafter. *Arts.* 164—6.

96. *To determine the foci of a small pencil refracted obliquely through a prism, the axis of the pencil passing in a principal plane of the prism.*

We will suppose the pencil to pass very near the edge of the prism, in order to examine the effect of the oblique refrac-

tions on the pencil independently of its length of path within the prism.



Let Q be the origin of a small pencil whose axis is obliquely refracted in a principal plane of the prism in direction QAS. Let Q_1, Q_2 be the primary and secondary foci after the first refraction, q_1, q_2 those at emergence,

$$AQ = u, \quad Aq_1 = v_1, \quad Aq_2 = v_2,$$

ϕ, ϕ' the angles of incidence and refraction at the 1st surface,

ψ, ψ' .... emergence and incidence .... 2nd ....

From the first refraction,

$$AQ_1 = \frac{\mu \cos^2 \phi'}{\cos^2 \phi} u.$$

Now the pencil emanating in the primary plane from Q_1 diverges in that plane after refraction from q_1: hence if we suppose its course reversed, a pencil converging to q_1 will converge in the primary plane to Q_1 after refraction into the prism;

$$\therefore AQ_1 = \frac{\mu \cos^2 \psi'}{\cos^2 \psi} v_1;$$

$$\therefore v_1 = \frac{\cos^2 \phi' . \cos^2 \psi}{\cos^2 \phi . \cos^2 \psi'} u \text{ .... (i).}$$

Similarly $\qquad AQ_2 = \mu u, \quad AQ_2 = \mu v_2;$

$$\therefore v_2 = u \text{ .... (ii).}$$

These results (i), (ii), combined with the relations

$\sin \phi = \mu \sin \phi'$, $\sin \psi = \mu \sin \psi'$, and $\phi' + \psi' = i =$ the refracting angle of the prism,—are sufficient to determine the position of the foci.

COR. 1. From (i), (ii), we may obtain

$$\frac{v_1}{v_2} = \frac{\cos^2 \phi' \cos^2 \psi}{\cos^2 \phi \cos^2 \psi'}, \text{ which is } = \frac{(\mu^2 - 1) \tan^2 \phi + \mu^2}{(\mu^2 - 1) \tan^2 \psi + \mu^2},$$

a result which shews that (μ being > 1) the primary or secondary focus is the nearer of the two to the prism, according as ϕ is < or > ψ.

COR. 2. If the pencil at emergence diverge from a *point*, i.e. if $v_1 = v_2$—we must have

$$\frac{\cos^2\phi'}{\cos^2\phi} \cdot \frac{\cos^2\psi}{\cos^2\psi'} = 1, \text{ and } \therefore \phi = \psi,$$

which result is also the condition of minimum deviation. In this case the emergent pencil diverges from a point at the same distance as its origin from the edge of the prism, and the degree of divergence remains unaltered.

This result is of importance in Newton's experiment hereafter described.

Successive Refractions at Spherical Surfaces.

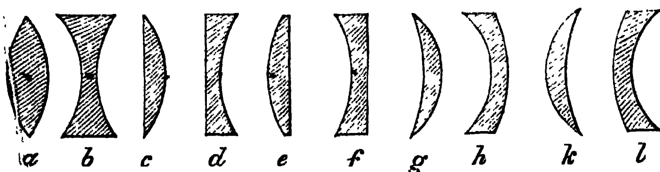
97. *Def.* A *lens* is a portion of a refracting medium bounded by two surfaces of revolution which have a common axis, called *the axis of the lens.*

Obs. The bounding surfaces of a lens will, unless the contrary be expressed, be considered spherical—under which denomination plane surfaces are included as a particular case when the radius of the sphere is infinite.

If the surfaces of revolution do not meet, the additional surface which is required as a boundary to the lens round the edge, will be a cylindrical one, having its axis coincident with that of the lens.

98. Lenses are distinguished by different names, according to the nature of their surfaces.

Thus in the figure:—



a is a double convex lens,
b ... double concave,
c ... convexo-plane,
d ... concavo-plane,
e ... plano-convex,
f ... plano-concave,
$\left.\begin{matrix} g \\ h \end{matrix}\right\}$ ... convexo-concave,
$\left.\begin{matrix} k \\ l \end{matrix}\right\}$ ... concavo-convex,

the forms g and k, in which the concave surface is less curved than the convex, are also known by the name of a *meniscus*.

In these figures, light is as usual supposed to come from the right, and the order of the terms which are combined to form the designation of any lens is that in which it is incident on the two surfaces.

Thus c and e are the same lens, but it is convexo-plane in the former case because light is incident first on the convex and then on the plane surface,—plano-convex in the latter case, because light is first incident on the plane and then on the convex surface.

It may be convenient to speak of the surfaces in the order in which light passes through them, as the *anterior* and *posterior* surfaces.

Obs. A pencil is said to be *directly* refracted through a lens, when the refraction at each surface is direct.

99. *To find the geometrical focus of a pencil after direct refraction through a lens, the thickness of which is neglected.*

Let Q be the origin of a pencil whose axis QAB passes directly through a lens, the thickness of which may be neglected, F_1 and F the geometrical foci of the pencil after one refraction and at emergence.

B A F Q F_1

Let $AQ = u$, $BF = v$, r, s the radii of the first and second surfaces

of the lens respectively, lines being regarded as positive when measured in a direction opposite to that of the incident light.

From the refraction at the first surface

$$\frac{\mu}{AF_1} - \frac{1}{u} = \frac{\mu - 1}{r}. \quad \text{(Art. 27)} \ldots\ldots\ldots\ldots\ldots \text{(i)}.$$

Now a pencil converging to F would have F_1 for its geometrical focus after refraction at B, (Art. 79);

$$\therefore \frac{\mu}{BF_1} - \frac{1}{v} = \frac{\mu - 1}{s} \quad .\text{(ii)};$$

therefore, if the thickness be neglected, or AF_1 regarded as $= BF_1$, we get

$$\frac{1}{v} - \frac{1}{u} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{s}\right),$$

which result determines the position of F.

Note. The points Q and F are called conjugate foci, with respect to the lens.

100. Cor. 1. If t be the thickness of the lens at the axis, since

$$t = BF_1 - AF_1,$$

we might from (i) and (ii) obtain

$$\frac{1}{\frac{1}{v} + \frac{\mu - 1}{s}} - \frac{1}{\frac{1}{u} + \frac{\mu - 1}{r}} = \frac{t}{\mu},$$

a relation which is strictly accurate, and which may be used when the thickness is too considerable to allow of any of its powers being neglected.

Cor. 2. If the thickness be sensible, but small, and $= t$, since

$$\frac{\mu}{BF_1} = \frac{\mu}{AF_1 + t} = \frac{\mu}{AF_1\left(1 + \frac{t}{AF_1}\right)} = \frac{\mu}{AF_1}\left(1 - \frac{t}{AF_1}\right) \text{ nearly,}$$

equation (ii) gives

$$\frac{\mu}{AF_1} - \frac{\mu t}{(AF_1)^2} - \frac{1}{v} = \frac{\mu - 1}{s};$$

$$\therefore \frac{1}{v} - \frac{1}{u} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{s}\right) - \frac{\mu t}{(AF_1)^2}$$

$$= (\mu - 1)\left(\frac{1}{r} - \frac{1}{s}\right) - \frac{t}{\mu}\left(\frac{1}{u} + \frac{\mu - 1}{r}\right)^2;$$

or the effect of the thickness is to remove the point F farther from A by a distance equal to

$$\frac{t}{\mu}\left(\frac{1}{u} + \frac{\mu - 1}{r}\right)^2 . v^2.$$

101. *Def.* The geometrical focus of a pencil of parallel rays refracted directly through a lens is called the *principal focus* of the lens,—and the distance of this point from the posterior surface of the lens is the *focal length* of the lens.

The focal length of a lens is commonly denoted by the symbol f. Hence we have

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{s}\right.$$

This is the expression for the focal length on the supposition that the thickness is neglected; if f' denote the focal length when the thickness is not neglected, we shall obtain from Art. 100, Cor. 1—(remembering that in this case $u = \infty$, $v = f'$),

$$\frac{1}{\frac{1}{f'} + \frac{\mu - 1}{s}} - \frac{1}{\frac{\mu - 1}{r}} = \frac{t}{\mu},$$

and $$\frac{1}{f} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{s}\right),$$

$$\text{whence } \frac{1}{f'} = \frac{1}{f} - \frac{\frac{\mu-1}{\mu} \cdot \frac{t}{r}}{\frac{r}{\mu-1} + \frac{t}{\mu}} \quad \text{.........................} \quad \text{(iii)},$$

a relation between f and f'.

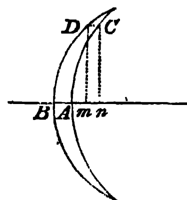
If we neglect the square of $\frac{t}{r}$, then the approximation deduced from (iii), gives

$$f' = f + \left(\frac{\mu-1}{r}\right)^2 \cdot \frac{tf^2}{\mu};$$

which accords with the result of Art. 100, Cor. 2.

Note. It will appear from the preceding, that when the thickness of a lens is not neglected the focal length does not remain the same when the order of the surfaces at which the refractions take place is reversed.

102. We will next examine in what cases the focal length of a lens is positive or negative.



Let $\tau = CD$ the thickness of a lens measured parallel to the axis at a distance $Cn = y$ from the axis, $t = AB$ the value of τ when $y = 0$, i.e. at the axis; r, s the radii of the first and second surfaces.

Then $\qquad An = \frac{y^2}{2r}$, very nearly,

$$Bm = \frac{y^2}{2s},$$

and $\qquad Bm + \tau = t + An;$

$$\text{i. e. } \tau - t = An - Bm = \frac{y^2}{2} \cdot \left(\frac{1}{r} - \frac{1}{s}\right) = \frac{y^2}{2(\mu-1) \cdot f}.$$

Hence f is positive or negative according as τ is $>$ or $<$ t, i.e. according as the lens is thinnest or thickest at the axis.

Thus lenses may be divided into two classes distinguished by the *sign* of the focal length.

A lens whose focal length is *positive* is called a *concave* lens;—one whose focal length is *negative* is called a *convex* lens.

Lenses may be constructed of an infinite variety of forms so as to have the same focal length—since there are two disposable quantities r, s, and only one relation connecting them, viz.

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{s}\right)$$

103. From a discussion of the equation

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f},$$

due regard being had to the algebraic signs of the symbols u, v, f, the student will have little difficulty in deducing the following inferences:—

(i) *Concave Lens.* A *divergent* pencil incident directly on a concave lens *diverges* after refraction. A *convergent* pencil consists at emergence of *diverging*, *parallel* or *converging* rays, according as its point of convergence is at a distance from the lens *greater than*, *equal to*, or *less than* the focal length of the lens.

(ii) *Convex Lens.* A *divergent* pencil incident directly on a convex lens consists at emergence of *diverging*, *parallel* or *converging* rays, according as its origin is at a distance from the lens *less than*, *equal to*, or *greater than* the focal length of the lens. A *convergent* pencil *converges* after refraction.

104. *Def.* The reciprocal $\left(\frac{1}{f}\right)$ of the focal length of a lens is called the *power* of the lens.

Since $\frac{1}{r}$, $\frac{1}{s}$ measure the *curvatures* of the surfaces of the lens, we see that the *power* of a lens is proportional to the difference of the curvatures of the two surfaces.

105. A sphere may be regarded as a lens, and the formula for the transmission of a direct pencil through it

deduced from Art. 100, Cor. 1; as however it is a case of some importance, we will obtain the formula independently. It will be convenient to measure lines from the centre of the sphere.

To find the geometrical focus of a pencil of rays after direct refraction through a sphere.

Let p be the distance from the centre of a refracting sphere of the origin of a pencil whose axis is refracted directly through the sphere, q_1, q the distances of the geometrical foci of the pencil from the centre after refraction at the first and second surfaces respectively—lines being considered positive when measured from the centre in a direction contrary to that of the incident pencil; r the radius of the sphere.

Then from refraction at the first surface,

$$\frac{1}{q_1} - \frac{\mu}{p} = -\frac{\mu - 1}{r}, \text{ (Art. 29) } \ldots\ldots\ldots\ldots\ldots \text{(i)},$$

and from refraction at the second surface, if the course of the pencil be supposed reversed, remembering that the radius is negative in this case,

$$\frac{1}{q_1} - \frac{\mu}{q} = \frac{\mu - 1}{r};$$

$$\therefore \frac{\mu}{q} - \frac{\mu}{p} = -2 \cdot \frac{\mu - 1}{r}, \text{ or } \frac{1}{q} - \frac{1}{p} = -2\frac{\mu - 1}{\mu r},$$

which determines the position of the geometrical focus of the emergent pencil.

106. *Def.* The *focal length* of a refracting sphere is the distance from the centre of the geometrical focus of a pencil of parallel rays after direct refraction through the sphere.

Hence if we write f for the focal length, $\frac{1}{f} = -2\frac{\mu - 1}{\mu r}$.

$$\text{and } \frac{1}{q} - \frac{1}{p} = \frac{1}{f}.$$

In a similar way the *focal length* of a *hemisphere*, measured from the *centre* of the spherical surface, would be $= -\dfrac{\mu r}{\mu - 1}$, if the pencil is incident *first* on the *plane* surface;

but $= -\dfrac{r}{\mu(\mu - 1)}$, if the pencil is incident *first* on the *spherical* surface.

107. *To trace the relative change of position of the conjugate foci of a pencil refracted directly through a thin lens.*

Suppose the lens *convex*, and let f be the *numerical* value of the focal length, then the relation connecting the positions of the conjugate foci is $\dfrac{1}{v} - \dfrac{1}{u} = -\dfrac{1}{f}$, where the lines u, v are taken positive to the right;—the direction from which light is supposed to proceed.

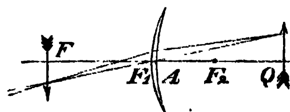
It is obvious from the formula that u and v must increase and decrease together, *algebraically*; i.e. Q, F always move in the same direction.

Take two points F_2, F_1 on the anterior and posterior sides of the lens so that $AF_2 = AF_1 = f$.

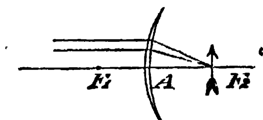
(i) Suppose $u = \infty$, i.e. the incident rays parallel, then $v = -f$, and F coincides with F_1 the principal focus, on the posterior side of the lens.

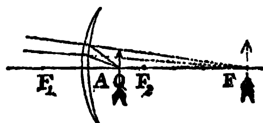


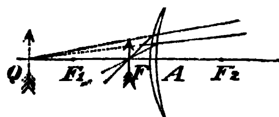
(ii) As Q moves from an infinite distance up to F_2, F also moves off to the left, and when Q coincides with F_2, F has gone off to an infinite distance and the emergent rays are parallel.



(iii) When Q moves from F_2 up to A, F appears on the positive side of A, and moves up towards A, the emergent pencil being divergent and the focus F a *virtual* one,—and Q, F arrive at A simultaneously.





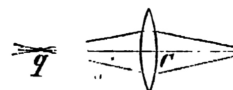


(iv) When u becomes negative or the incident pencil is convergent, the emergent pencil is convergent,—and as Q moves off from A to an infinite distance, F moves from A up to F_1.

Similarly the change of relative position may be traced when the lens is *concave*.

108. *Method of determining the focal length of a convex lens practically.*

Let a small bright object Q be placed further from the lens than its principal focus, q the real image of Q on the opposite side of the lens,



$$CQ = u,$$

f = numerical focal length,

$$Qq = x, \text{ then } \frac{1}{Cq} + \frac{1}{CQ} = \frac{1}{f},$$

$$\text{or } \frac{1}{x-u} + \frac{1}{u} = \frac{1}{f};$$

$$\therefore fx = ux - u^2;$$

$$\therefore u = \frac{x \pm \sqrt{\{x(x-4f)\}}}{2}.$$

It appears from this result that the least positive value of x is $4f$: hence if the image of Q formed at q be received on a screen, and the lens and screen be moved backward and forward till the distance of the image from Q is the least possible, the focal length of the lens—without reference to algebraic sign—is one-fourth of this distance.

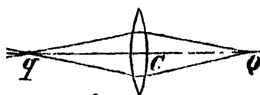
If the lens be concave, let it be placed in contact with a convex lens whose focal length f' is such that the focal length F of the combination may be negative, the axes of the two lenses being coincident.

Then if f' and F be each determined in the preceding manner f is to be found from the formula (*algebraic*)

$$\frac{1}{F} = \frac{1}{f'} + \frac{1}{f}, \quad \text{(Art. 114)}$$

$$\text{or } \frac{1}{f} = \frac{1}{F} - \frac{1}{f'}.$$

108*. *Cylindrical Lenses.* If the figure represent a section of such a lens perpendicular to its length, the image of a bright point Q will be extended into a line—slightly curved—and parallel to the length of the lens.

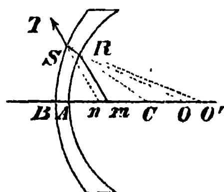


Such lenses are occasionally employed,—for the purpose of extending the image of a bright *point* into a *line* of light,—as by Mr Huggins in his observation of stellar spectra.

Centre of a Lens.

109. *Def.* The *centre* of a lens is a point in its axis where a line joining the extremities of two parallel radii of its two surfaces cuts the axis.

Let OAB be the axis; O, O' the centres of the first and second surfaces; R, S points in those surfaces where the normals RO, SO' are parallel.



Join SR and produce it, if necessary, to cut the axis in C,—C is the *centre* of the lens.

Let r, s be the radii of the surfaces, $AB = t$.

Then by similar triangles $\frac{CO}{CO'} = \frac{r}{s}$,

$$\text{or } \frac{r - AC}{s - t - AC} = \frac{r}{s};$$

whence $AC = \frac{rt}{s - r}$, which determines the position of C.

The position of C evidently depends only upon the *form* of the lens.

By observing the value of the expression for AC in different cases, we shall arrive at the following results respecting the position of C.

(i) If the curvatures of the two surfaces of the lens are in opposite directions (as in a, b, fig. Art. 98), the *centre* lies *within* the lens.

(ii) If one surface be plane (as in c, d, e, f) the centre lies on the *curved* surface.

(iii) If the curvatures are in the same direction (as g, h, k, l) the *centre* lies without the lens—on the *convex* side in g, k—on the *concave* side in h, l.

(iv) If the thickness of the lens be neglected, i.e. $t = 0$, then $AC = 0$, and the centre of the lens may be regarded as coinciding with the point A.

Obs. C is the *centre of similitude* of the circles AR, BS.

Focal Centres.

110. *Def.* The ultimate positions of the points on the axis where the incident and emergent ray cuts the axis, when the direction of the ray between the two refractions passes through the centre of the lens, are called the *focal centres* of the lens.

To find their position.

Let $mRST$ be the course of a ray, the direction of RS passing through C the centre; m, n the focal centres; r, s the radii of the first and second surfaces. (*Fig. Art.* 109.)

Then in the limit when the ray coincides with the axis, we have from refraction at the first surface

$$\frac{\mu}{AC} - \frac{1}{\quad} = \frac{\mu - 1}{\quad}$$

and from refraction at the second surface

$$\frac{\mu}{BC} - \frac{1}{Bn} = \frac{\mu - 1}{s},$$

and $AC = \dfrac{rt}{s - r}$, whence we get

$$Am = \frac{rt}{s(s - r - t) + t}, \; Bn = \frac{st}{\mu(s - r - t) + t},$$

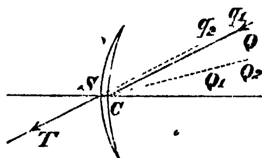
which determine the positions of m and n.

111. If a ray be refracted through a lens in such a manner that its direction between the two refractions passes through C the centre of the lens, its directions at incidence and at emergence from the lens will be parallel to one another. For if R and S be the points of incidence and emergence the surfaces at these points being parallel, the case is the same as that of a ray refracted through a plate whose surfaces touch the lens at R and S.

When a pencil is refracted obliquely through a lens there will be an important difference produced, according as the pencil is refracted *centrically* or *excentrically*, i.e. according as the direction of its axis between the two refractions does or does not pass through the centre of the lens.

112. *When a small pencil is obliquely and centrically refracted through a thin lens, to find the distances of the foci of the emergent pencil from the centre of the lens.*

Let Q be the origin of a small pencil whose axis $QCST$ is refracted obliquely and centrically through a lens,—C being the point where the axis of the lens meets its first surface, which point is the centre of the lens if the thickness of the lens be neglected (Art. 109). Let Q_1, Q_2 be the primary and secondary foci of the pencil after one refraction, q_1, q_2 the primary and secondary foci of the emergent pencil.



Let $CQ = u$, $Sq_1 = v_1$, $Sq_2 = v_2$, r, s the radii of the first and second surfaces of the lens, ϕ, ϕ' the angles of incidence and refraction of the axis QC at C, and consequently the angles of emergence and incidence at S.

From refraction at the first surface

$$\frac{\mu \cos^2 \phi'}{CQ_1} - \frac{\cos^2 \phi}{u} = \frac{\mu \cos \phi' - \cos \phi}{r}, \text{ (Art. 68).}$$

Now the pencil emanating in the primary plane from Q_1 diverges in that plane after the second refraction from q_1;

hence if we suppose its course reversed, a pencil converging to q_1 will in the primary plane converge to Q_1 after refraction into the lens;

$$\therefore \frac{\mu \cos^2 \phi'}{CQ_1} - \frac{\cos^2 \phi}{v_1} = \frac{\mu \cos \phi' - \cos \phi}{s},$$

the points C, S being regarded as coincident;

$$\therefore \frac{1}{v_1} - \frac{1}{u} = \frac{\mu \cos \phi' - \cos \phi}{\cos^2 \phi} \cdot \left(\frac{1}{r} - \frac{1}{s}\right) \ldots\ldots\ldots \text{(i)}.$$

Similarly $$\frac{\mu}{CQ_2} - \frac{1}{u} = \frac{\mu \cos \phi' - \cos \phi}{r},$$

$$\frac{\mu}{CQ_2} - \frac{1}{v_2} = \frac{\mu \cos \phi' - \cos \phi}{s};$$

$$\therefore \frac{1}{v_2} - \frac{1}{u} = (\mu \cos \phi' - \cos \phi) \left(\frac{1}{r} - \frac{1}{s}\right) \ldots\ldots\ldots \text{(ii)}.$$

Equations (i), (ii) determine the distances of the foci of the emergent pencil from S or C.

113. Cor. 1. The positions of the foci of the refracted pencil being thus known, the investigation of (Art. 70) gives the magnitude and position of the circle of least confusion.

Cor. 2. If ϕ be so small that its square may be neglected,

$$\frac{1}{v_1} = \frac{1}{v_2} = \frac{1}{u} + (\mu - 1) \left(\frac{1}{r} - \frac{1}{s}\right),$$

the same as for a direct pencil.

Hence a pencil refracted centrically through a lens at small obliquity approximately converges to or diverges from a point at the same distance from the lens as the geometrical focus of a direct pencil with an origin at the same distance.

Cor. 3. If we make a closer approximation neglecting powers of ϕ above the third and putting $\phi = \mu\phi'$, we shall obtain

$$\frac{1}{f} = (\mu - 1) \left(\frac{1}{r} - \frac{1}{s}\right),$$

$$\frac{1}{v_1} - \frac{1}{u} = \frac{1}{f}\left\{1 + \left(1 + \frac{1}{2\mu}\right)\phi^2\right.$$

$$\frac{1}{v_2} - \frac{1}{u} = \frac{1}{f}\left\{1 + \frac{1}{2\mu}\phi^2\right.$$

For rays parallel at incidence, i.e. $u = \infty$,

$$v_1 = f\left\{1 - \left(1 + \frac{1}{2\mu}\right)\phi^2\right\},$$

$$v_2 = f\left\{1 - \frac{1}{2\mu}\phi^2\right\}:$$

and if f' be the distance from the lens of the circle of least confusion,

$$f' = \frac{v_1 v_2 (1 + \cos\phi)}{v_1 \cos\phi + v_2} = f\left\{1 - \frac{\mu + 1}{2\mu} \cdot \phi^2\right\}.$$

Hence, *practically* the focal length of a lens is diminished—or the *power* increased—by inclining it slightly to the line of sight.

COR. 4. If Δ be the distance of the circle of least confusion from the centre of the lens, the incident rays not being parallel,

$$\frac{1}{\Delta} = \frac{v_1 \cos\phi + v_2}{v_1 v_2 (1 + \cos\phi)} = \frac{\dfrac{\cos\phi}{v_2} + \dfrac{1}{v_1}}{1 + \cos\phi}$$

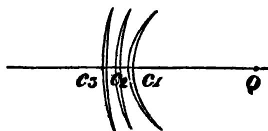
$$= \frac{1}{v_2} + \left(\frac{1}{v_1} - \frac{1}{v_2}\right)\left(\frac{1}{1 + \cos\phi}\right.$$

$$= \frac{1}{u} + \frac{1}{f} + \left(1 + \frac{1}{\mu}\right)\frac{\phi^2}{2f}.$$

Combinations of Lenses.

114. *To find the geometrical focus of a pencil of rays after direct refraction through a series of lenses in contact, whose axes are coincident.*

Let Q be the origin of a pencil whose axis $QC_1C_2...$ is refracted directly through a series of n lenses on a common axis, their centres being C_1, C_2...



Let u be the distance of Q from C_1; v_1, v_2 the distances from C_1, C_2... respectively of the geometrical foci of the pencil after refraction through the first, second,... lens:—f_1, f_2... the focal lengths of the successive lenses;—lines being considered positive when measured in a direction contrary to that of the incident pencil.

Then if the thickness of each lens be neglected,

$$\left.\begin{aligned} \frac{1}{v_1} - \frac{1}{u} &= \frac{1}{f_1} \\ \frac{1}{v_2} - \frac{1}{v_1} &= \frac{1}{f_2} \\ &\dots \\ \frac{1}{v_n} - \frac{1}{v_{n-1}} &= \frac{1}{f_n} \end{aligned}\right\} \quad (A);$$

$$\therefore \frac{1}{v_n} - \frac{1}{u} = \frac{1}{f_1} + \frac{1}{f_2} + \dots + \frac{1}{f_n} = \Sigma\left(\frac{1}{f}\right) = \frac{1}{F}, \text{ suppose,}$$

which determines the position of the geometrical focus of the emergent pencil;—and shews that the *power* of a combination of lenses in contact is equal to the *sum of the powers of the several lenses.*

COR. If the lenses be separated by finite intervals

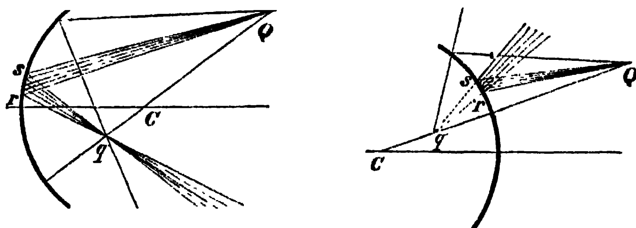
$$a_1, a_2, ..a_{n-1},$$

we have in place of equations (A), the following:

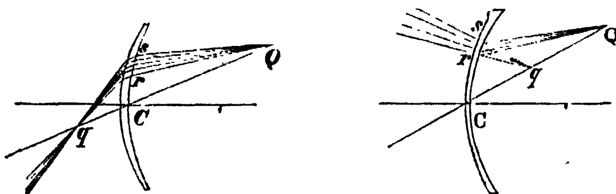
$$\begin{aligned} \frac{1}{v_1} - \frac{1}{u} &= \frac{1}{f_1} \\ \frac{1}{v_2} - \frac{1}{v_1 + a_1} &= \frac{1}{f_2} \\ \dots\dots\dots\dots &= .. \\ \frac{1}{v_n} - \frac{1}{v_{n-1} + a_{n-1}} &= \frac{1}{f_n} \end{aligned}$$

By eliminating $v_1, v_2 \ldots v_{n-1}$ between these n equations v_n is determined.

115. If QC be the axis of an oblique pencil reflected at a spherical mirror or refracted centrically through a lens, we may generally—in the cases which occur in Optical Instruments where the obliquity is small—regard,



(i) the reflected pencil as converging to or diverging from a point q,—Q, q being conjugate foci on the line QCq which passes through C the centre of the spherical surface;



(ii) the refracted pencil as diverging from or converging to a point q which lies on the line QCq passing through C the centre of the lens,—the point q being taken to be the position of the circle of least confusion—or else determined from the equation $\frac{1}{Cq} - \frac{1}{CQ} = \frac{1}{f}$, which will generally be sufficiently approximate.

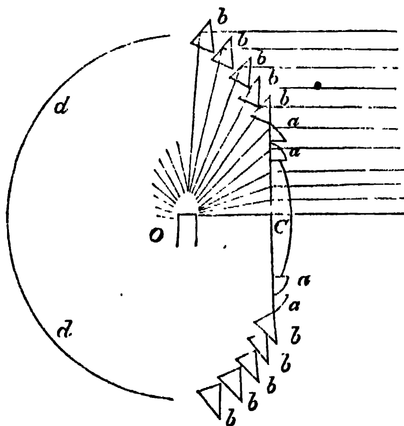
When only a small part of such a pencil exists—as the small excentrical pencil Qrs—we may still regard this excentrical portion as diverging from or converging to the same point Q after reflection or refraction.

The student will bear in mind that this supposition is only approximately true; but it is sufficiently so for the purpose of description and explanation of pencils passing through Optical Instruments, and we shall make frequent use of it hereafter.

116. The following construction for determining the position of q for an excentrical pencil is accurate to the same degree of approximation as is above supposed.

Let F be the principal focus of rays,—proceeding from right to left suppose,—either in the case of a lens or mirror; Q the origin of the pencil, QR a ray parallel to the axis of the lens or mirror and incident at R, QCq the ray which passes through C the centre of the lens or mirror,—join RF,—then, if we neglect aberration, RF or FR is the direction of the ray QR after reflexion or refraction,—and on the same approximate supposition q will be the point in which QC and RF,—produced if necessary,—intersect each other.

117. The forms of lenses now commonly used in *Lighthouses* will perhaps be understood from the following brief description.



The figure represents a section of the lens by a plane through its axis, with respect to which axis the lens is symmetrical.

The central part C is the same as the section of a convex lens, the light O being at its principal focus,—so that the rays at emergence from C are parallel to OC. $a, a \ldots$ are sections of prisms,—the front surfaces of which are however somewhat convex,—so that the rays emerge from them parallel to OC, and $b, b \ldots$ another set of prisms from which the rays emerge parallel to OC after internal reflexion; d, d is a section of a polished spherical reflector whose centre is O.

(i) If we suppose the figure to revolve about OC, a *horizontal* axis, there will be formed an *annular lens*,—and the beams transmitted through the prisms a, a, would be hollow cylindrical shells surrounding the central beam transmitted through the lens C. In this form of lens the extreme prisms $b, b \ldots$ are omitted, and when two or more such lenses are fixed in a polygonal frame which is made to revolve about a vertical axis passing through O,—the cylindrical beams transmitted through these annular lenses sweep the horizon and produce a *revolving* or *periodic* light.

(ii) If we suppose the figure to revolve through any given angle about a vertical axis through O, there will be formed a cylindrical lens,—and the light transmitted through the several parts of it will form a horizontal beam in the form of a sector of a circle embracing the part of the horizon within which it is intended to be seen, and constituting a *fixed* or *permanent* light.

The rays diverging from O in the direction opposite to that of the lens are returned by means of the reflector d, d—and from the fact that by means of the lens, prisms, and reflector, the rays which diverge from O in every direction are rendered serviceable, this arrangement is called *holophotal*.

The light O consists in general of a set of Argand lamps.

For further information on this subject the student may consult the article *Lighthouses*, in the *Encyclopædia Britannica*.

CHAPTER VI.

REFRACTION THROUGH MEDIA OF VARYING DENSITY:—REFLEXION AND REFRACTION AT A SURFACE IN ANY MANNER.

118. WHEN a ray traverses a medium the density of which varies continuously from point to point, we may regard the value of μ,—the absolute refractive index,—at any point as a function of the position of the point; and the equation $\mu = C$, a constant quantity, would belong to a surface at every point of which the absolute refractive index is the same,—and the form of this surface will indicate the law of stratification of the medium. By varying this constant continuously we should obtain the consecutive surfaces of equal refractive index in the medium; and if $\mu, \mu + d\mu$ be the *absolute* indices of refraction for two consecutive strata, the *relative* index from the former of the two into the latter will be $\frac{\mu + d\mu}{\mu}$. (Art. 83, 84.)

When the change of density and therefore of refractive index is continuous, the ordinary law of refraction will lead to a differential equation, the solution of which will enable us to determine the path of the ray.

119. Suppose a ray passing from a stratum A into another B,—μ, $\mu + d\mu$ the absolute refractive indices of A and B,—ϕ, $\phi + \delta$ the angles of incidence and refraction at the common surface of A, B:—then we have (Art. 10),

$$\sin \phi = \frac{\mu + d\mu}{\mu} \cdot \sin (\phi + \delta),$$

now if $d\mu$ be a small quantity, δ will be a small quantity, and we shall have, neglecting squares and products of small quantities,

$$0 = \frac{d\mu}{\mu} \sin \phi + \delta \cos \phi;$$

$$\therefore \ \delta = -\frac{d\mu}{\mu} \tan \phi;$$

i.e. when the difference of refractive index is small, the *deviation* varies as *the tangent of the angle of incidence.*

Obs. The coefficient $\frac{d\mu}{\mu}$ in accordance with the results of experiment is assumed to be proportional to the difference of density of the strata, when that difference is small.

If μ be given as a function of certain co-ordinates, we may regard $\mu = c$ a constant, as the equation to a surface of equal density, or a surface of equal absolute refractive power; and if we suppose c to vary slowly and continuously we shall obtain a series of consecutive surfaces of equal density or of equal absolute refractive power, and in this way the law of variation of density of a *stratified medium* may be expressed. For example, if μ involve only one of the rectangular co-ordinates, as x, the medium will be stratified in planes parallel to the plane of yz;—if μ involve only the radius vector r, the medium will be arranged in consecutive spherical surfaces of equal density; if μ involve x and y only, the stratification will be symmetrical with respect to the plane of x, y.

120. The following examples will illustrate the application of the preceding principles in finding the path of a ray in a medium of varying density.

To find the path of a ray in a medium, the density at any point of which varies as its distance from a fixed plane.

The ray will pass in one plane—which take as plane of xy; and the axis of x perpendicular to the fixed plane.

λx the density at a distance x from the fixed plane.

Then (Art. 119, *Obs.*)

$$\frac{d\mu}{\mu} \propto \lambda \, . \, dx = \kappa \, . \, dx, \text{ suppose},$$

$$\therefore \; \mu = A \, . \, \epsilon^{\kappa x},$$

when $x = 0$, $\mu = 1$; $\therefore A = 1$, and $\mu = \epsilon^{\kappa x}$.

Again, at any point, if ϕ be the angle which the tangent to the path of the ray makes with axis of x,—which angle is the angle of incidence,—

$$\sin\phi = \frac{\mu + d\mu}{\mu} . \sin(\phi + d\phi),$$

whence $$d\phi . \cot\phi = -\frac{d\mu}{\mu} = -\kappa . dx;$$

$$\therefore \quad \log\sin\phi = \log C - \kappa x;$$

$$\therefore \quad \sin\phi = \sin\alpha . \epsilon^{-\kappa x}, \qquad \text{if } \phi = \alpha \text{ when } x = 0.$$

And $$\frac{dy}{dx} = \tan\phi;$$

$$\therefore \frac{dy}{dx} = \frac{\sin\alpha . \epsilon^{-\kappa x}}{\sqrt{(1 - \sin^2\alpha . \epsilon^{-2\kappa x})}}.$$

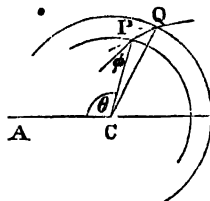
The integral of which gives

$$\sin(\alpha - \kappa y) = \sin\alpha . \epsilon^{-\kappa x}, \text{ if } y = 0 \text{ when } x = 0.$$

The curve has an asymptote $y = \dfrac{\alpha}{\kappa}$.

121. *A ray passes through a medium, the value of μ at any point of which is a function of r the distance from a fixed point; to find the equation to the path of the ray.*

The medium is stratified in spherical surfaces concentric with the fixed point C.



Let $ACP = \theta$, $CP = r$ be co-ordinates of any point P of the path of the ray, which will be in one plane passing through C.

$$\mu = f(r),$$

ϕ, $\phi + d\phi$ the angles which the radius-vector makes with the tangent to the path at two contiguous points P, Q, then $\phi + d\phi + d\theta$ is the angle of refraction at P.

Whence $$\sin\phi = \frac{\mu + d\mu}{\mu}\sin(\phi + d\phi + d\theta);$$

therefore neglecting squares and products of small quantities,

$$\frac{d\mu}{\mu}\sin\phi + (d\phi + d\theta)\cos\phi = 0 \text{...........} \quad \text{(i)},$$

and remembering that

$$\cot\phi = \frac{dr}{r\,.\,d\theta} \quad \text{.....................} \quad \text{(ii)},$$

this becomes

$$\frac{d\mu}{\mu} + \cot\phi\,.\,d\phi + \frac{dr}{r} = 0\,;$$

integrating we get $\quad \mu r \sin\phi = C, \quad$ a constant.
Or if p be the perpendicular from C on the tangent the result can be expressed simply by

$$\mu p = C \quad \text{.......................} \text{(iii)},$$

the equivalent form in $r\theta$ is

$$\frac{d\theta}{dr} = \frac{C}{r\sqrt{(\mu^2 r^2 - C^2)}}.$$

Note. The form (iii) might have been obtained more readily,—for if ϕ' be the angle of refraction at P, $\frac{\mu'}{\mu}$ the refractive index,

$$\sin\phi = \frac{\mu'}{\mu}\sin\phi'\,;\ \therefore\ \mu' r \sin\phi' = \mu r \sin\phi\,;$$

$$\therefore\ \ \mu' p' = \mu p\,;\ \therefore\ d\,(\mu p) = 0, \text{ or } \mu p = C \text{ a constant.}$$

Ex. 1. Suppose $\mu \propto \frac{1}{r}$, the path is an equiangular spiral.

Ex. 2. Suppose $\mu \propto \frac{1}{\sqrt{r}}$, the path is a parabola, focus C.

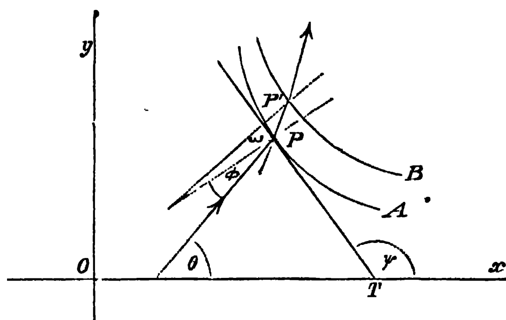
Ex. 3. Suppose $\mu \propto \frac{1}{\sqrt{r^2 - a^2}}$, the path is an epicycloid or a hypocycloid.

Ex. 4. Suppose $\mu \propto r^{\frac{3}{2}}$, the path is $\left(\frac{a}{r}\right)^{\frac{5}{2}} = \sin\frac{5}{2}(\alpha - \theta)$.

122. *A ray is propagated through a medium of variable density which is symmetrical with respect to the plane of xy —(which is also the plane of the ray),—the differential equation to the path is*

$$\frac{\frac{d^2y}{dx^2}}{1+\left(\frac{dy}{dx}\right)^2}=\frac{1}{\mu}\left(\frac{d\mu}{dy}-\frac{d\mu}{dx}\frac{dy}{dx}\right),$$

where $\mu=f(xy)$ *is the refractive index at any point* xy, *and* $\frac{d\mu}{dx}, \frac{d\mu}{dy}$ *are the partial differential coefficients of* μ.



Let PT be the tangent to the curve $\mu=C$ at a point P where the ray enters the stratum A,—P' the point where the ray enters the contiguous stratum B, $\psi=\angle PTx$,

then
$$\frac{d\mu}{dx}+\frac{d\mu}{dy}\tan\psi=0.$$

Again, if ϕ be the angle which the ray at P makes with the normal—ω the angle which the normals to the surfaces $\mu=C$, $\mu=C+dC$ make with each other, these normals being drawn at P, P'; then taking xy to be the current coordinates of the path of the ray, we have $\omega=d\psi$ and the angle of refraction at $P=\phi+d\phi+d\psi$;

$$\therefore\ \sin\phi=\frac{\mu+d\mu}{\mu}\sin(\phi+d\phi+d\psi),$$

whence
$$0=\frac{d\mu}{\mu}\sin\phi+(d\phi+d\psi)\cos\phi.$$

Now $$d\mu = \frac{d\mu}{dx}.dx + \frac{d\mu}{dy}.dy,$$

$$\tan\theta = \frac{dy}{dx},\ \psi - \theta = \frac{\pi}{2} - \phi;\qquad \therefore d\phi + d\psi = d\theta;$$

$$\therefore d\theta.\cot\phi + \frac{1}{\mu}\left(\frac{d\mu}{dx}.dx + \frac{d\mu}{dy}.dy\right) = 0,$$

and $$d\theta = d.\left(\tan^{-1}\frac{dy}{dx}\right),\qquad \therefore \frac{d\theta}{dx} = \frac{\frac{d^2y}{dx^2}}{1+\left(\frac{dy}{dx}\right)^2},$$

and $$\cot\phi = \tan(\psi - \theta) = \frac{\tan\psi - \tan\theta}{1+\tan\psi\tan\theta}$$

$$= -\frac{\frac{d\mu}{dx} + \frac{d\mu}{dy}.\frac{dy}{dx}}{\frac{d\mu}{dy} - \frac{d\mu}{dx}.\frac{dy}{dx}};$$

whence $$\frac{\frac{d^2y}{dx^2}}{1+\left(\frac{dy}{dx}\right)^2} = \frac{1}{\mu}\left(\frac{d\mu}{dy} - \frac{d\mu}{dx}.\frac{dy}{dx}\right),$$

which is the differential equation to the path of the ray.

123. We may easily obtain the same result from the principle that when a ray passes through any medium then $\Sigma(\mu.ds)$ = a minimum, or

$$\int \mu\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}.dx = \text{a minimum}.$$

With the usual notation of the *Calculus of Variations* since μ is a function of x and y, $=\phi(xy)$ suppose,

$$V = \phi(xy)\sqrt{(1+p^2)} \text{ and } N - \frac{dP}{dx} = 0 \ldots\ldots\ldots \text{(i)}.$$

Now $$N = \frac{d\mu}{dy}\sqrt{(1+p^2)},\ P = \frac{\mu p}{\sqrt{(1+p^2)}};$$

therefore (i) gives

$$\sqrt{(1+p^2)}\cdot\frac{d\mu}{dy} = \frac{d}{dx}\left\{\mu\frac{p}{\sqrt{(1+p^2)}}\right\}$$

$$= \left(\frac{d\mu}{dx} + \frac{d\mu}{dy}p\right)\frac{p}{\sqrt{(1+p^2)}} + \mu\frac{\frac{dp}{dx}}{(1+p^2)^{\frac{3}{2}}};$$

$$\therefore \frac{\frac{dp}{dx}}{1+p^2} = \frac{1}{\mu}\left(\frac{d\mu}{dy} - \frac{d\mu}{dx}\cdot\frac{dy}{dx}\right),$$

the same as before.

If ρ be the radius of curvature at the point xy of the path the preceding result can be put in the form

$$\frac{\mu}{\rho} = \frac{d\mu}{dx}\cdot\frac{dy}{ds} - \frac{d\mu}{dy}\cdot\frac{dx}{ds}.$$

124. If $\mu = \phi\,(xyz)$ be the absolute refractive index in the most general case of a medium of varying density, from the principle that $\Sigma\,(\mu ds) =$ a minimum, we shall obtain by the methods of the Calculus of Variations the relations

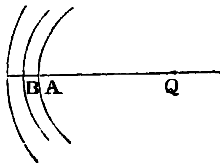
$$\frac{d\mu}{dx} - \frac{d}{ds}\left(\mu\frac{dx}{ds}\right) = 0,$$

$$\frac{d\mu}{dy} - \frac{d}{ds}\left(\mu\frac{dy}{ds}\right) = 0,$$

$$\frac{d\mu}{dz} - \frac{d}{ds}\left(\mu\frac{dz}{ds}\right) = 0,$$

any two of which, when integrable, will give the path of the ray.

125. Suppose a series of media bounded by spherical surfaces whose centres lie in the same straight line,—Q the origin of a ray which passes very nearly centrically through the series;



$AQ = u = \frac{1}{U}$, v_1, v_2... the distances from A, B... of the conjugate focus after refraction at A, B...

t_1 t_2... the thicknesses AB... between successive surfaces;

μ_0 μ_1 μ_2... the indices of refraction of successive media.

Then index of refraction at $A = \frac{\mu_1}{\mu_0}$, at $B = \frac{\mu_2}{\mu_1}$,...

and we have

$$\frac{\mu_1}{v_1} - \frac{\mu_0}{u} = \frac{\mu_1 - \mu_0}{r_1} \text{ .... (i),}$$

$$\frac{\mu_2}{v_2} - \frac{\mu_1}{u_1} = \frac{\mu_2 - \mu_1}{r_2} \text{ .... (ii),}$$

where $u_1 = v_1 + t_1$

.... = ...

and u, v_1, v_2...r_1 r_2... are measured positive to the right, t_1 ... positive to the left.

When the thicknesses of the media are finite, we may obtain the focus of the central rays at emergence by successive elimination between (i), (ii),...

If the thicknesses are indefinitely small, for convenience write

$$v_1 = \frac{1}{V},\ v_2 = \frac{1}{V + dV},\ r_1 = \frac{1}{R},\ r_2 = \frac{1}{R + dR},$$

$$u_1 = v_1 + t_1 = \frac{1}{V} + dx = \frac{1 + V.dx}{V};$$

$\mu_2 = \mu_1 + d\mu_1 = \mu + d\mu$, suppressing the accent,

(ii) becomes

$$(\mu + d\mu)\ (V + dV) - \mu \frac{V}{1 + Vdx} = d\mu\,(R + dR);$$

$$\text{or } V.d\mu + \mu.dV + \mu V^2.dx = R.d\mu,$$

neglecting squares, &c. of small quantities;

$$\text{or } dV + (V - R)\frac{d\mu}{\mu} + V^2 dx = 0,$$

a differential equation for determining the focus of emergent rays.

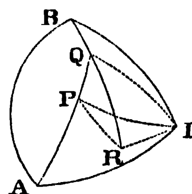
Two equations connecting the general values of R, μ, x are required to be given before the integration can be attempted.

The above would apply to a lens of variable density—like the *crystalline lens* of the Eye.

See Herschel, Article *Light*, before referred to;—Gauss, *Dioptrische Untersuchungen*, Gottingen.

126. In Articles (77, 91), we have shewn how to determine the direction of a ray after reflexion or refraction at two surfaces in succession, when the ray passes in a principal plane,—in order to complete this part of the subject we proceed to shew how to determine the direction of a ray after two reflexions or refractions when it passes in any manner whatever,—and first for reflexion.

To determine the direction of a ray, reflected at two plane surfaces in any manner.



Let radii of a sphere be drawn parallel to the ray *at incidence on the first surface*, after the *first* and after the *second* reflexion,—their directions being contrary to that of the course of the ray in each case,—and let them meet the surface of the sphere in P, Q, R, respectively. Also let radii be drawn parallel to the normals to the first and second reflecting surface,—in each case towards that side of the surface at which the reflexion takes place,—and meeting the sphere in A, B. Also let a radius be drawn parallel to the line of intersection of the surfaces and meeting the sphere in I. Draw the great circles APQ, BQR, AB, PI, &c.

Let $\phi = AP = \pi - AQ =$ the $\angle$ of incidence on the first surface,

$\psi = BQ = \pi - BR$ .... second ......

$D = PR =$ the deviation,

$i = \pi - AB =$ the inclination of the surfaces,

$\left.\begin{array}{l}\theta_1 = PAB \\ \theta_2 = QBA\end{array}\right\}$ the angles which the planes of first and second reflexion make with the principal plane AB, i.e. the plane perpendicular to each surface,

$\omega = AQB =$ the angle between the planes of first and second reflexion.

Then observing that $PQ = \pi - 2\phi$, $QR = \pi - 2\psi$, we get

$$\frac{\sin\omega}{\sin i} = \frac{\sin\theta_1}{\sin\psi} = \frac{\sin\theta_2}{\sin\phi} \quad \text{...............(i), (ii)},$$

$$\cos i = \cos\psi\cos\phi - \sin\psi\sin\phi\cos\omega \quad \text{.........(iii)},$$

$$\cos D = \cos 2\phi\cos 2\psi - \sin 2\phi\sin 2\psi\cos\omega \quad \text{...(iv)}.$$

These four equations connecting the seven quantities

$$\phi,\ \psi,\ \theta_1,\ \theta_2,\ i,\ \omega,\ D,$$

are sufficient to determine the circumstances of the ray, when any three of them are given.

COR. 1. We can easily shew that $IP = IR$, for

$\cos IP = \sin PA \cos PAI$, since $AI = \frac{\pi}{2}$ in the $\Delta\ PAI$

$= \sin PA . \sin PAB$

$= \sin QA . \sin QAB$

$= \sin QB . \sin QBA = \sin QB . \cos QBI\ (= \cos QI)$

$= \sin RB . \cos QBI = \cos RI.$

Hence, $IP = IQ = IR$, that is, the inclination of the ray to the line of intersection of the mirrors remains unchanged.

COR. 2. Further, since $PIQ = \pi - 2AIP$, $QIR = \pi - 2BIQ$;

$\therefore PIR = QIR - PIQ = 2(AIP - BIQ)$

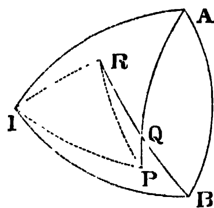
$= 2\{(AIR + PIR) - (\pi - BIR)\} = 2\{(AIR + BIR) - \pi + PIR\}$

$= 2\{(\pi - i) - \pi + PIR\}$; $\therefore PIR = 2i.$

That is, planes drawn through the line of intersection of the surfaces parallel to the direction of the ray before the first and after the second reflexion include an angle which is double the angle between the reflecting surfaces.

127. *To find the direction of a ray after being refracted at two plane surfaces in any manner.*

Let radii of a sphere be drawn parallel to the direction of the ray at incidence on the first surface, after the first and after the second refraction,—their directions being contrary to that of the pencil in each case, and let them meet the surface of the sphere in P, Q, R.



Also let radii parallel to the normals to the first and second surfaces—drawn on that side of each surface on which the ray is incident—meet the sphere in A, B: and a radius parallel to the intersection of the two surfaces, in I.

Let

$\left.\begin{array}{l} PA = \phi \\ QA = \phi' \end{array}\right\}$ the angles of incidence and refraction at 1st surface,

$\left.\begin{array}{l} QB = \psi' \\ RB = \psi \end{array}\right\}$ .... 2nd surface,

$AB = i$ the angle between the two surfaces,

$PAB = \theta_1$, $QBA = \theta_2$ the angles which the planes of *first* and *second* refraction make with the principal plane, i.e. the plane perpendicular to each surface,

$\omega = AQB =$ the angle between the planes of *first* and *second* refraction,

$\mu\mu'$ the indices of refraction from the *first* to the *second*, and from the *second* to the *third* medium,

$D = PR$ the deviation of the ray;—then we have the relations

$$\frac{\sin \omega}{\sin i} = \frac{\sin \theta_1}{\sin \psi'} = \frac{\sin \theta_2}{\sin \phi'},$$

$$\cos i = \cos \phi' \cos \psi' + \sin \phi' \sin \psi' \cos \omega,$$

$$\cos D = \cos(\phi - \phi')\cos(\psi - \psi') + \sin(\phi - \phi')\sin(\psi - \psi')\cos \omega$$

and

$$\sin \phi = \mu \sin \phi', \quad \sin \psi' = \mu' \sin \psi.$$

These equations are sufficient to determine the circumstances of the transmission of the ray, when the quantities given render the problem determinate.

COR. If the first and third medium are the same, the case becomes that of refraction through a prism, and $\mu . \mu' = 1$. In this case we can shew that $IP = IR$.

For $$\cos IP = \sin PA . \cos PAI = \sin PA . \sin QAB$$
$$= \mu . \sin QA . \sin QAB$$
$$= \mu . \sin BQ . \sin QBA$$
$$= \sin BR . \sin QBA$$
$$= \sin BR . \cos RBI$$
$$= \cos IR;$$

whence $IP = IR$, that is, when a ray is refracted in any manner through a prism, its directions before the first and after the second refraction are equally inclined to the edge of the prism.

Note. Several analytical results defining the course of a ray reflected or refracted under different circumstances, will be found among the problems on this Chapter.

CHAPTER VII.

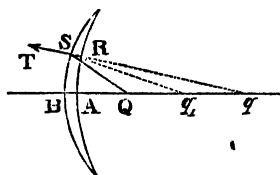
SPHERICAL ABERRATION OF LENSES;—
EXCENTRICAL PENCILS.

128. In Chapter V. we have neglected the aberration arising from the spherical form of the surfaces of the lens through which a pencil passes. In the construction of lenses for telescopes, &c., it becomes a question of great importance to ascertain what particular *forms* can be given to a lens or a combination of lenses so as to destroy the aberration of a pencil passing through the instrument,—or, if this cannot be done, to reduce it within the narrowest possible limits.

The formulæ and calculations to which this enquiry leads are very tedious and intricate, and we only propose in this place to give one or two of the more simple investigations bearing on this subject, as an indication of the nature of the processes to be carried out,—referring the Student who has leisure and inclination to pursue the subject to works where it is more fully discussed.

See a Paper by Sir John Herschel, *On the Aberrations of Compound Lenses and Object Glasses. Phil. Trans.*, 1821. A Paper by the Astronomer Royal, *On Spherical Aberration of the Eye-pieces of Telescopes. Camb. Phil. Trans.* Vol. III. *Astron. Notices,* Dec. 1862, Vol. XXIII. p. 69. Art. *Light,* in *Encycl. Metrop.* Coddington's *Optics.*

129. *When a pencil is refracted directly through a lens, to find the point where the direction of any ray cuts the axis after refraction—and the aberration of the pencil.*



Let Q be the origin of a pencil whose axis QAB is refracted directly through a lens of inconsiderable thickness. Let $QRST$ be the course of any ray whose directions after one refraction and at emergence cut the axis in q_1 and q respectively. Let $AQ = u$, $Aq = v'$, r, s the radii of the first and second surfaces of the lens,—lines being accounted positive when measured in a direction contrary to that of the incident pencil, $AR = y$.

Now from the first refraction,

$$\frac{\mu}{Aq_1} - \frac{1}{u} = \frac{\mu - 1}{r} + \frac{\mu - 1}{\mu^2}\left(\frac{1}{r} - \frac{1}{u}\right)^2\left(\frac{1}{r} - \frac{\mu + 1}{u}\right)\frac{y^2}{2}, \text{ (Art. 53)...(i).}$$

If we suppose the course of the pencil reversed, a ray of a pencil converging to q after refraction at BS cuts the axis in q_1. Hence, neglecting the thickness of the lens so that $BS = AR = y$, we obtain

$$\frac{\mu}{Aq_1} - \frac{1}{v'} = \frac{\mu - 1}{s} + \frac{\mu - 1}{\mu^2}\left(\frac{1}{s} - \frac{1}{v'}\right)^2\left(\frac{1}{s} - \frac{\mu + 1}{v'}\right)\frac{y^2}{2} \text{ ...... (ii);}$$

$$\therefore \frac{1}{v'} - \frac{1}{u} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{s}\right)$$

$$+ \frac{\mu - 1}{\mu^2}\left\{\left(\frac{1}{r} - \frac{1}{u}\right)^2\left(\frac{1}{r} - \frac{\mu + 1}{u}\right) - \left(\frac{1}{s} - \frac{1}{v'}\right)^2\left(\frac{1}{s} - \frac{\mu + 1}{v'}\right)\right\}\frac{y^2}{2}.$$

In the coefficient of y^2 we may use for v' its first approximate value v given by the equation

$$\frac{1}{v} - \frac{1}{u} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{s}\right), \text{ (Art. 49);}$$

$$\frac{1}{v'} - \frac{1}{u} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{s}\right)$$

$$+ \frac{\mu - 1}{\mu^2}\left\{\left(\frac{1}{r} - \frac{1}{u}\right)^2\left(\frac{1}{r} - \frac{\mu + 1}{u}\right) - \left(\frac{1}{s} - \frac{1}{v}\right)^2\left(\frac{1}{s} - \frac{\mu + 1}{v}\right)\right\}\frac{y}{2}$$

which determines the position of the point q,

and the *aberration* of the ray $qS = v' - v$

$$= -\frac{\mu-1}{\mu^2}\left\{\left(\frac{1}{r}-\frac{1}{u}\right)^2\left(\frac{1}{r}-\frac{\mu+1}{u}\right)-\left(\frac{1}{s}-\frac{1}{v}\right)^2\left(\frac{1}{s}-\frac{\mu+1}{v}\right)\right\}\frac{v^2y^2}{2}.$$

Note. The position and magnitude of the least circle of aberration in this or any other case of combined direct refractions is given by Art. 56.

130. *To investigate the form of a lens of given focal length in order that the aberration of a given direct pencil of parallel rays may be the least possible.*

The aberration of a pencil of parallel rays refracted directly through a lens the radii of whose surfaces are r, and s,—putting $u = \infty$ and $v = f$ in the expression for the aberration in the preceding article—

$$\propto \frac{1}{r^3}-\left(\frac{1}{s}-\frac{1}{f}\right)^2\left(\frac{1}{s}-\frac{\mu+1}{f}\right) \text{..............} \text{(i)},$$

where $(\mu-1)\left(\frac{1}{r}-\frac{1}{s}\right)=\frac{1}{f}$, a given quantity......... (ii).

The former expression is therefore to be made a minimum by the variation of r and s consistently with the latter condition.

If we differentiate (i) with respect to $\frac{1}{r}$, and observe that by (ii) the differential coefficient of $\frac{1}{s}$ is $= 1$, we get

$$0 = \frac{3}{r^2}-\left(\frac{1}{s}-\frac{1}{f}\right)\left(\frac{3}{s}-\frac{2\mu+3}{f}\right)$$

$$= 3\left\{\frac{1}{s}+\frac{1}{(\mu-1)f}\right\}^2-\left(\frac{1}{s}-\frac{1}{f}\right)\left(\frac{3}{s}-\frac{2\mu+3}{f}\right)$$

$$\frac{\mu}{(\mu-1)^2f}\left\{\frac{2(\mu-1)(\mu+2)}{s}-\frac{2\mu^2-\mu-4}{f}\right\};$$

$$\therefore \frac{s}{f} = \frac{2(\mu-1)(\mu+2)}{2\mu^2-\mu-4}$$

and $\frac{s}{r} = 1 + \frac{s}{(\mu-1)f} = 1 + \frac{2(\mu+2)}{2\mu^2-\mu-4} = \frac{\mu(2\mu+1)}{2\mu^2-\mu-4}$ ... (iii),

also the above value of s makes the second differential coefficient of (i) positive, so that the relation of the radii obtained in (iii) makes the aberration a *minimum*.

130*. If we call the minimum aberration δf we shall obtain

$$\delta f = -\frac{\mu(4\mu-1)}{8(\mu-1)^2(\mu+2)} \cdot \frac{y^2}{f}.$$

The aberration tends to become less as μ increases—but it remains considerable for all substances known in nature.

If $\mu = 2$, as in *zircon*, $\delta f = -\frac{7}{16}\frac{y^2}{f}$,

If $\mu = 2{\cdot}5$, as in *diamond*, $\delta f = -\frac{5}{18}\frac{y^2}{f}$.

See *infra*, Art. 246.

See a short Paper by Lord Rayleigh "'On the minimum "aberration of a single lens for parallel rays." *Proceedings of Cambridge Philosophical Society*, Vol. III., part VIII., p. 373.

131. Ex. 1. Suppose $\mu = 1{\cdot}5$, then $\frac{s}{r} = -6$,

$$s = -\frac{7}{2}f,\ r = \frac{7}{12}f,$$

and the value of the minimum aberration will be found by substitution in the expression of Article 129 to be

$$= -\frac{15}{14}\frac{y^2}{f}.$$

If the lens be a convex one, that is f negative, and the ratio of $s : r = 6 : 1$, *this* lens is known by the name of a *crossed lens.*

Ex. 2. If $2\mu^2 - \mu - 4 = 0$;

i.e. $\mu = 1{\cdot}6861$, then $\frac{s}{r} = \infty$,

and the most advantageous form of the lens for collecting all the rays into a *real* focus is convexo-plane, having the anterior surface convex. This value of μ is nearly the refractive index of several precious stones and the more refractive glasses.

132. *To find the form of a lens of given focal length in order that the aberration of a direct pencil diverging from a point at a given distance may be the least possible.*

The focal length of the lens being given, the value of v for a proposed value of u is known. Hence we must have

$$\left(\frac{1}{r}-\frac{1}{u}\right)^2\left(\frac{1}{r}-\frac{\mu+1}{u}\right)-\left(\frac{1}{s}-\frac{1}{v}\right)^2\left(\frac{1}{s}-\frac{\mu-1}{v}\right)=\text{a minimum},$$

whilst $$(\mu-1)\left(\frac{1}{r}-\frac{1}{s}\right)=\frac{1}{f}=\frac{1}{v}-\frac{1}{u}.$$

In consistence with these two latter conditions, let us assume

$$\frac{1}{v}=\frac{\alpha+1}{2f},\quad \frac{1}{u}=\frac{\alpha-1}{2f},$$

also $$\frac{1}{r}=\frac{x+1}{2(\mu-1)f},\quad \frac{1}{s}=\frac{x-1}{2(\mu-1)f},$$

so that x is the only variable quantity,—we get by substitution,

$$\left(\frac{1}{r}-\frac{1}{u}\right)^2\left(\frac{1}{r}-\frac{\mu+1}{u}\right)$$

$$=\frac{1}{8f^3(\mu-1)^3}\{x-(\mu-1)\alpha+\mu\}^2\{x-(\mu^2-1)\alpha+\mu^2\}\ \ldots\ \text{(i)}.$$

Now we pass from $\left.\begin{matrix}\frac{1}{r}\text{ to }-\frac{1}{s}\\ \frac{1}{u}\text{ to }-\frac{1}{v}\end{matrix}\right\}$ by changing the signs of $\left\{\begin{matrix}x\\ \alpha\end{matrix}\right.$;

$$-\left(\frac{1}{s}-\frac{1}{v}\right)^2\left(\frac{1}{s}-\frac{\mu+1}{v}\right)$$

$$=-\frac{1}{8f^3(\mu-1)^3}\{x-(\mu-1)\alpha-\mu\}^2\{x-(\mu^2-1)\alpha-\mu^2\}\ \ldots\ \text{(ii)}.$$

In adding (i) and (ii), terms of odd dimension in x and a disappear, and the sum becomes

$$\frac{\mu}{4(\mu-1)^2 f^3}\left\{\frac{\mu+2}{\mu-1}x^2 - 4(\mu+1)ax + (\mu-1)(3\mu+2)a^2 + \frac{\mu^3}{\mu-1}\right\},$$

which is to be a *minimum* by the variation of x;

$$\therefore 0 = \frac{\mu+2}{\mu-1}x - 2(\mu+1)a;$$

$$\therefore x = \frac{2(\mu^2-1)}{\mu+2}a,$$

which determines r and s, and consequently the *form* of the lens. The ratio of the radii $= \frac{s}{r} = \frac{x+1}{x-1}$.

COR. If the incident pencil consist of parallel rays,

$$u = \infty \text{ and } a = 1,\ \frac{s}{r} = \frac{\mu(2\mu+1)}{2\mu^2-\mu-4},$$

the same result as was obtained in the previous article.

133. *Obs.* The expression obtained above involves the quantity a depending upon the distance from which the incident pencil diverges, so that it appears the best form of lens for diminishing the aberration will vary for different distances of the origin of the pencil.

If the aberration for rays parallel at incidence on a compound lens of given focal length—consisting of several thin lenses in contact—be examined, it will consist of a series of terms similar to that in Art. (129), one term for each lens, and the condition that the aberration shall vanish will lead to an equation involving more than one unknown quantity—and consequently admitting of an unlimited number of solutions.

As an instance of an aplanatic combination of two lenses of glass ($\mu = 1{\cdot}5$), we may mention one calculated by Sir John Herschel.

First lens $\begin{cases} r = -5{\cdot}833 \\ s = 35 \end{cases}$ $f = -10,$

second lens in contact with the first,

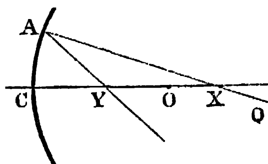
$$r = - \ \ 3{\cdot}688, \text{ or } -2{\cdot}054,$$
$$s = - \ \ 6{\cdot}291, \text{ or } -8{\cdot}128,$$
$$f = -17{\cdot}829, \text{ or } -5{\cdot}497.$$

The first lens of the combination is a *crossed lens*, the second a *meniscus*—the focal length of the combination is $-6{\cdot}407$, or $-3{\cdot}474$ according as the first or second of the two sets of values for r, s, f in the second lens, are taken.

134. *Excentrical Pencils.*

The mode of determining the form and course of *direct pencils* and of *oblique centrical pencils* when reflected or refracted has been sufficiently explained in the preceding pages. All pencils which do not belong to one or other of these classes are *excentrical*. For the investigations requisite to determine accurately the direction and form of such pencils the Student is referred to Coddington's *Optics*, Part I. We shall here give an approximate mode of defining the direction of the axis of an excentrical pencil which will be of service hereafter—the axis of the pencil being supposed in each case to lie in one plane with the axis of the reflecting or refracting surface—this being the ordinary case which occurs in Optical Instruments.

135. *To find the course of the axis of a pencil after excentrical reflexion at a spherical surface.*



Let QXA be the axis of a pencil incident at A on a spherical reflecting surface of which O is the centre of the surface and C the centre of the face, AY the axis of the reflected pencil, so that X, Y are the points where the axis of the *incident* and *reflected* pencil cuts CYX the axis of the reflecting surface. Let $CO = r$, $CX = b$, $CY = c'$, lines being considered positive when measured from C in the direction more nearly opposite to that of the incident pencil.

Also let $AC = y$, $\angle AYC = \eta$, $\angle AXC = \epsilon$.

Then, as in (Art. 21) $\frac{XA}{YA} = \frac{XO}{YO}$,

whence if powers of y above the first be neglected, and c represent the first approximate value of c', since in that case X, Y may be regarded as conjugate foci, (Art. 22),

$$\frac{1}{c} + \frac{1}{b} = \frac{2}{r},$$

and
$$\frac{\tan \eta}{\tan \epsilon} = \frac{b}{c};$$

equations giving c and η,—quantities which define the direction and position of the axis after reflexion.

136. *To find the course of the axis of a pencil after excentrical refraction through a thin lens.*

Let b, c' be the distances from the centre of the lens of points where the axis of the pencil cuts the axis of the lens before and after refraction, r, s the radii of the first and second surfaces of the lens, lines being considered positive when measured in the direction more nearly opposite to that of the incident pencil. Also let y be the distance of the point of incidence of the axis from the centre of the lens; ϵ, η the inclinations of the axis of the pencil to the axis of the lens before and after refraction, f the focal length of the lens.

Then by an investigation similar to that of Art. (129), if powers of y above the first be neglected, and c represent the first approximate value of c',

$$\frac{1}{c'} - \frac{1}{b} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{s}\right) + Ay^2,$$

$$\frac{\tan \eta}{\tan \epsilon} = \frac{b}{c'};$$

or, approximately, $\frac{1}{c} - \frac{1}{b} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{s}\right)$,

$$\frac{\tan \eta}{\tan \epsilon} \quad b.$$

which results give c and η.

137. *To find the course of the axis of a pencil after excentrical refraction through a series of lenses which have a common axis.*

Let the axis of the pencil cut the axis of the lenses before and after refraction through the first lens at distances b_1, c_1 from its centre, and let f_1 be the focal length of the lens. Let the same letters b, c, f with suffixes 2, 3 ... denote similar quantities relative to the second, third, ... lens, and if the thickness of the lenses be neglected, as a first approximation we have the system of equations

$$\frac{1}{c_1}-\frac{1}{b_1}=\frac{1}{f_1},\quad \frac{1}{c_2}-\frac{1}{b_2}=\frac{1}{f_2}\ldots\frac{1}{c_n}-\frac{1}{b_n}=\frac{1}{f_n}\text{ (Art. 114)},$$

whence c_n may at last be found.

Also if ϵ be the inclination of the axis of the pencil to that of the lenses before incidence, $\eta_1, \eta_2, \ldots \eta_n$ its inclinations after refraction through the several lenses, we have

$$\frac{\tan\eta_1}{\tan\epsilon}=\frac{b_1}{c_1},\quad \frac{\tan\eta_2}{\tan\eta_1}=\frac{b_2}{c_2}\quad\ldots\quad\frac{\tan\eta_n}{\tan\eta_{n-1}}$$

whence
$$\frac{\tan\eta_n}{\tan\epsilon}=\frac{b_1b_2\ldots b_n}{c_1c_2\ldots c_n};$$

which results give c_n and η_n.

Obs. In each of the cases of the preceding three articles if the calculation were carried to a second approximation, we should have results of the form

$$\frac{1}{c'}=\frac{1}{c}+Ay^2,$$

$$\frac{\tan\eta}{\tan\epsilon}=\frac{b}{c}(1+By^2),$$

where the quantities Ay^2, By^2 are terms introduced by the *aberrations* as calculated in previous articles.

Cor. We should find that $B \propto \frac{1}{f^2}$: and if b be large and therefore $c=f$ nearly, we should have

$$\tan\eta \propto \frac{1}{f}\left(1+B'\frac{y^2}{f^2}\right).$$

138. *Equivalent Lens.*

Def. A lens is *equivalent* to a system of lenses on the same axis when an excentric pencil after refraction through it is inclined at the same angle to the axis as if it had been refracted through the system of lenses—the single lens having the position of that lens of the system on which the pencil is first incident.

If F be the focal length of a lens equivalent to the lenses in the last proposition, and if the pencil after refraction through it cut the axis at an angle η at a distance c, then using first approximations, we have

$$\frac{\tan\eta}{\tan\epsilon} = \frac{b_1}{c}, \quad \frac{1}{c} - \frac{1}{b_1} = \frac{1}{F} \text{ (Art. 136)},$$

$$\frac{b_1}{F} + 1 = \frac{\tan\eta}{\tan\epsilon} = \frac{\tan\eta_n}{\tan\epsilon} = \frac{b_1 b_2 \dots b_n}{c_1 c_2 \dots} \text{ (Art. 137)},$$

$$\therefore \frac{1}{F} = \frac{b_2 b_3 \dots b_n}{c_1 c_2 c_3 \dots c_n} - \frac{1}{b}$$

If as is generally the case in the eye-glasses of telescopes b_1 be very large, and $\frac{1}{b_1}$ may be neglected, we have

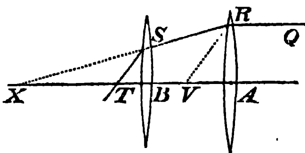
$$F = \frac{c_1 c_2 \dots c_n}{b_2 b_3 \dots b_n}.$$

Note. See a paper *On Equivalent Lenses* by *R. Pendlebury, M.A.—Messenger of Mathematics*, Vol. VII., p. 129, Jan. 1878.

139. We will give an independent investigation of the following example of an equivalent lens.

To find the focal length (F) *of a lens* equivalent *to a combination of two lenses—(focal lengths f_1, f_2)—on the same axis at a distance* (a) *from each other.*

Suppose the two lenses to be convex and their thickness neglected. Let a ray QR parallel to the axis be refracted by the first and second lens in directions RS and ST.



Then if we draw RV parallel to ST, AV will represent F the focal length of the equivalent

lens, i.e. AV is the focal length of a lens which will produce the same *deviation* in a ray incident parallel to the axis, as the combination of the two lenses produces.

Now $$AX = f_1; \therefore BX = f_1 - a.$$

Also $$\frac{1}{BT} - \frac{1}{BX} = \frac{1}{f_2};$$

$$\therefore \frac{1}{BT} = \frac{1}{f_2} + \frac{1}{f_1 - a} = \frac{f_1 + f_2 - a}{f_2(f_1 - a)}.$$

Also $AV = F$, and by similar triangles AVR, BTS,

$$\frac{F}{BT} = \frac{AR}{BS} = \text{similarly } \frac{AX}{BX};$$

$$\therefore \frac{1}{F} = \frac{BX}{AX.BT} = \frac{(f_1 - a)}{f_1} \frac{f_1 + f_2 - a}{f_2(f_1 - a)},$$

$$= \frac{f_1 + f_2 - a}{f_1 f_2};$$

$$\therefore F = \frac{f_1 f_2}{f_1 + f_2 - a}, \text{ the focal length required.}$$

Obs. In the above each of the *three* lenses is treated as a convex lens, and f_1, f_2, F as their *numerical* focal lengths—if they were treated as concave lenses, we should obtain

$$F = \frac{f_1 f_2}{f_1 + f_2 + a},$$

which is *algebraically* true in all cases of the combination supposed in the statement.

140. *Ex.* 1. Suppose $f_1 = f_2$ and $a = \frac{2}{3}f_1$, then $F = \frac{3}{4}f_1$.

This combination represents *Ramsden's Eye-piece*—two convex lenses of equal focal length, placed on the same axis at a distance from each other equal to $\frac{2}{3}$ the focal length of either. The lenses have generally one surface plane, the other convex,—and the convex surfaces turned towards each other.

Ex. 2. Suppose $f_1 = 3f_2$, $a = f_1 - f_2$; $\therefore \frac{f_1 + f_2}{2} = a$.

$$\text{Then } F = \tfrac{3}{2} f_2 = \tfrac{1}{2} f_1.$$

This combination represents *Huyghens' Eye-piece*,—the focal length of the first, or *field*, lens being three times that of the second, or *eye*, lens,—each being convex and the distance between them equal to the difference, or semi-sum, of their focal lengths. The field-glass is generally convexo-concave and the eye-glass convexo-plane.

CHAPTER VIII.

IMAGES AND CAUSTICS.

141. DEF. If a luminous body be placed before a reflecting or refracting surface, a pencil of rays will emanate from each point of the surface of the body, and will have after reflexion or refraction a geometrical focus or circle of least confusion, according as the incidence is direct or oblique. The locus of these foci or circles of least confusion for consecutive points of the object will form a superficial figure, more or less resembling the object and called its *geometrical image.*

If the rays be reflected or refracted more than once there will of course be an image formed after each reflexion or refraction.

The consecutive intersections of the reflected or refracted rays emanating from any given point of the object will form a caustic surface, and an eye in any suitable position will receive a small pencil of rays, the axis of which is a tangent to this caustic, and apparently diverging from the point of contact of this tangent with the caustic—or nearly so. The locus of these points of contact for consecutive points of the objects constitutes the *visible image*—the position of which, as well as its form and general resemblance to the object, will vary with every change of position of the eye. See the illustrations given in Chapter III. Art. 74. It will readily be seen that the difference between the *visible* and the *geometrical* image, both as regards position and form, will in general be greater, the greater the obliquity of the pencils by which the former is seen,—but when this obliquity is small and the extent of object viewed inconsiderable, the *visible* image will very nearly coincide with the *geometrical*,—and in such cases they may appreciably be regarded as coincident. This will be strictly the case at the centre of the field of view of a tele-

scope—and when the field of view is not large it may be regarded as true over its whole extent.

When we speak of *image* simply, we shall in general mean the *geometrical image*.

142. Def. If an image consist of points through which the light actually passes it is called *real*;—in other cases *virtual*.

Hence a screen placed in the position of an image will receive illumination only when the image is real.

A familiar instance of a *virtual* image is that formed by a common looking-glass of an object in front of it:—the image of an object under water is *virtual*. The images formed by the object-glass of an astronomical telescope—by the large mirror of a Gregorian Telescope—by a camera obscura—are *real*.

Further, a real visible object and its optical image differ in this respect—from the former, light emanates in every direction, and it can be seen in any direction, if nothing opaque is interposed between it and the eye,—an image can only be seen when the eye is placed in the pencil of rays which go to form it, or diverge from it. If however the image be received on a screen,—such as paper or roughened glass—the light constituting the image illuminates the screen like a picture, and it can be viewed by the eye as if it were a real object.

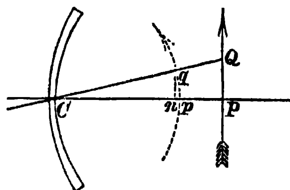
143. Def. An image is *erect* when corresponding points of the object and image are on the same side of the axis of the reflecting or refracting surface—*inverted*, when they are on opposite sides.

In the former case the corresponding parts of the object and image appear in the same relative directions in space—in the latter, the image is inverted as regards *up* and *down*, and also reversed as regards *right* and *left*. Considering the line of sight along the axis of the lens or mirror as the centre of the field of view, if the image were turned through 180° about this axis, the parts of the image would then be in the same relative position as the corresponding parts of the object.

The image of an object seen by reflexion at a *plane* mirror in a *vertical* position, suppose—is *erect* as regards *up* and *down*—but *reversed* as regards *right* and *left*.

144. *Geometrical image of a straight line placed before a lens,—cutting the axis of the lens at right angles.*

Let PQ be the object, C the centre of the lens. We will suppose the image formed by centrical pencils, and the size of the object PQ to be small, so that we may take the formula



$$\frac{1}{Cq} - \frac{1}{CQ} = \frac{1}{f} \dots\dots\dots (i),$$

to be that which connects the conjugate foci of any point, Art. (115).

$$CP = a, \quad Cq = r, \quad f = \text{focal length of the lens,}$$

then
$$CQ = \frac{a}{\cos\phi}, \text{ and (1) becomes}$$

$$\frac{1}{r} - \frac{\cos\phi}{a} = \frac{1}{f}, \text{ or } r = \frac{f}{1 + \frac{f}{a}\cos\phi}.$$

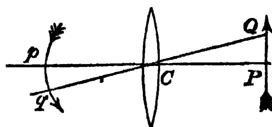
Hence the image near the vertex is a conic section of which C is one focus, latus rectum $= 2f$, and excentricity $= \frac{f}{a}$. It will be an ellipse, parabola, or hyperbola according as

$$a > = < f,$$

the radius of curvature at p the vertex of the image $= f$.

(i) Suppose the lens concave, f positive, $a > f$, the image is elliptic, virtual, and diminished.

(ii) Suppose the lens convex, f negative, $a > f$, the image is real, inverted, and elliptic,—C the further focus,—and magnified or diminished according as

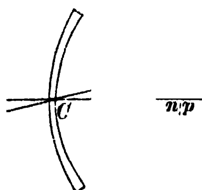


$$Cp > < CP,$$

i.e. as $\frac{af}{a-f} > < a$, or $a < > 2f$.

Obs. In a similar way the geometrical image of a straight line formed by reflexion at a spherical mirror may be examined, and will be found to be a conic section.

145. COR. 1. It may be useful to examine the more accurate value of the curvature at the vertex of the image, when the primary focus of an emergent pencil is taken to be the image of the corresponding point.



Then $Cq = v_1$, $CQ = u = \frac{a}{\cos\phi}$ connected by the formula

$$\frac{1}{v_1} - \frac{1}{u} = \frac{\mu\cos\phi' - \cos\phi}{\cos^2\phi\,.\,(\mu-1)f} \quad \text{(Art. 112)};$$

$$\therefore \frac{1}{v_1} = \frac{\cos\phi}{a} + \frac{\mu\cos\phi' - \cos\phi}{\cos^2\phi\,(\mu-1)f}.$$

Let $Cp = V$, then $\frac{1}{V} = \frac{1}{a} + \frac{1}{f}$,

also $Cn = v_1\cos\phi$, $qn = v_1\sin\phi$;

$$\therefore \rho = \text{rad. curv. at } p = \frac{1}{2}\,\text{Lt.}\,\frac{(qn)^2}{pn} = \frac{1}{2}\,\text{Lt.}\,\frac{v_1^2\sin^2\phi}{V - v_1\cos\phi}$$

$$= \frac{1}{2}\,\text{Lt.}\,\frac{v_1\sin^2\phi}{V\left\{\frac{\cos\phi}{a} + \frac{\mu\cos\phi' - \cos\phi}{\cos^2\phi\,(\mu-1)f}\right\} - \cos\phi}$$

$$= \frac{1}{2}\,\text{Lt.}\,\frac{v_1}{V}\cdot\frac{\sin^2\phi}{\frac{\cos\phi}{a} + \frac{\mu\cos\phi' - \cos\phi}{\cos^2\phi\,(\mu-1)f} - \cos\phi\left(\frac{1}{a} + \frac{1}{f}\right)}.$$

Now $\phi' = \frac{\phi}{\mu}$, nearly;—and Lt. $\frac{v_1}{V} = 1$ when $\phi = 0$;

$$\therefore\ \rho = \frac{f}{2}\,\text{Lt.}\left\{\frac{\sin^2\phi}{\dfrac{\mu\cos\phi' - \cos\phi}{\cos^2\phi\,(\mu-1)} - \cos\phi}\right\}$$

$$= \frac{f}{2}\,\text{Lt.}\frac{\phi}{\dfrac{\mu\left(1-\dfrac{\phi'^2}{2}\right) - \left(1-\dfrac{\phi^2}{2}\right)}{(\mu-1)(1-\phi^2)} - \left(1-\dfrac{\phi^2}{2}\right)}$$

$$= \frac{f}{\left(3+\dfrac{1}{\mu}\right)},\ \text{after reduction.}$$

146. Cor. 2. Suppose the image to be formed by the circles of least confusion.

Let $\Delta = Cq$ = distance from C to circle of least confusion

$$= \frac{v_1 v_2 (1+\cos\phi)}{v_1\cos\phi + v_2}\quad \text{(Art. 70)},$$

$qn = \Delta\sin\phi$, Cp = value of Cq when $\phi = 0$, and is

$$= \frac{1}{\dfrac{1}{a}+\dfrac{1}{f}};$$

$$\therefore\ 2\rho = \text{Lt.}\frac{(qn)^2}{pn} = \text{Lt.}\frac{\Delta^2\sin^2\phi}{Cp - Cn} = \text{Lt.}\frac{\Delta^2\sin^2\phi}{Cp - \Delta\cos\phi}$$

$$= \text{Lt.}\frac{\Delta}{Cp}\cdot\frac{\sin^2\phi}{\dfrac{1}{\Delta} - \dfrac{\cos\phi}{Cp}},\ \text{and}\ \frac{\Delta}{Cp} = 1\ \text{ultimately.}$$

$$\text{Also}\ \frac{1}{\Delta} = \frac{\cos\phi}{a} + \frac{1}{f} + \left(1+\frac{1}{\mu}\right)\frac{\phi^2}{2f}\ \text{(Art. 113, Cor. 4)},$$

$$\therefore\ 2\rho = \text{Lt.}\frac{\phi^2}{\dfrac{1-\cos\phi}{f} + \left(1+\dfrac{1}{\mu}\right)\dfrac{\phi^2}{2f}}$$

$$= f \cdot \frac{1}{1 + \frac{1}{2}\mu}; \qquad \frac{f}{2 + \frac{1}{\mu}},$$

$$\text{and } \therefore \text{ the curvature} = \frac{1}{\rho} = \left(2 + \frac{1}{\mu}\right)\frac{1}{f}.$$

Note. If the image be formed by centrical pencils refracted through a system of thin lenses of same substance in contact,—the curvature of the image at the vertex would be

$$\left(2 + \frac{1}{\mu}\right)\Sigma\left(\frac{1}{f}\right);$$

which, as we see, depends only on the *power* of the combination,—and not on the *forms* of the lenses or the *position* of the object.

The investigation of the curvature when the image is formed by excentrical pencils is more complicated—for which and other results for excentrical pencils the student is referred to Coddington's *Optics*.

Distortion.

147. The resemblance of an image to the object will seldom be perfect. If the image were exactly similar to the object, the ratio of the distance of two points of the image to the distance of the corresponding points of the object would be uniform for every combination of points that could be taken. When this is not the case, the image is *distorted.*

Let us consider the image formed by a lens or a spherical mirror. If the ratio of the distances of two points of the image and of the two corresponding points of the object—measured in each case *parallel to the axis* of the lens or mirror—be not constant for different points, there is a distortion in direction of the axis. This is commonly called *distortion of curvature* of the image.

The image discussed in Art. (144) suffers this kind of distortion, and this particular example will account for the apparent convexity of the field of view of an astronomical

telescope, since a *plane* surface in front of the object-glass would have for its image—at the principal focus of the object-glass—a surface concave towards the object-glass, and therefore *convex* towards the eye-glass; on which convex surface the objects contained in the original plane field observed, would appear to be distributed.

Again, let the points of the image and object be referred to polar co-ordinates (r, θ)—the radius vector r being measured perpendicular to the axis of the lens or mirror, and the $\angle\theta$ being the angle which r makes with a fixed plane passing through the same axis. Then if the ratio of the radii vectores of any point of the image and of the corresponding point of the object is not constant when different points are taken—there is *linear distortion* of the image.

And when the values of θ are not the same for pairs of corresponding points of the image and object,—there is *angular distortion.*

When an image is formed by *centrical* pencils, neither linear nor angular distortion will have any existence. When the image is formed by *excentrical* pencils—as in the case of the eye-glass of a telescope—there will be linear distortion in the image:—and if the eye be not on the axis of the lens, or if the surfaces of the lens be not surfaces of revolution, there will be angular distortion also.

Obs. In all cases of oblique reflexion or refraction, the image does not in general consist of an assemblage of *points*, but of circles of confusion overlapping one another, and it is therefore called *indistinct.*

This cause of imperfection of an image cannot be altogether obviated, and it will in general vary at different points of the image:—the magnitude of the circles of confusion at different points might be taken as a measure of the comparative indistinctness.

148. *Caustics.* 

In Chapter III. we have indicated the general method which must be adopted to find the caustic formed by reflexion

or refraction in any case, and we have there given some examples of their formation and of their use in enabling us to construct the visible image of an object. We will here give a few general theorems respecting caustics and a general explanation of the apparent deformation of an object seen by reflexion at a mirror,—or through a lens.

149. (i) Suppose an object is viewed through a convex lens. Let *QRSE* be the course of the axis of a pencil by which any point of it is seen. Then this pencil is deflected by the lens in the same way as it would be by passing through a prism whose faces coincide with the tangent planes to the surfaces of the lens at *R* and *S*. The edge of such a prism would be turned *from* the axis of the lens and the deviation of the pencil would be greater the greater the angle of the prism,—so that the angular displacement of points of the image from the axis of the lens would be greater than in proportion to the distance of the corresponding points of the object from the same axis. Hence the distortion of an object like fig. 1 would in character resemble that given in fig. 2.

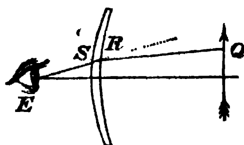


Fig. 1.

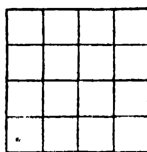


Fig. 2.

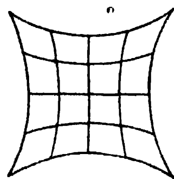
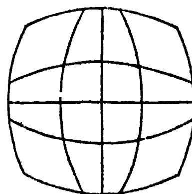


Fig. 3.

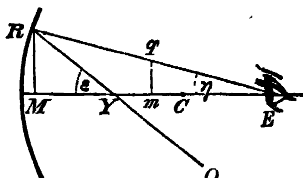


(ii) If an object be viewed through a concave lens, a similar mode of reasoning will shew that the distorted image of fig. 1 would resemble the appearance of fig. 3.

This accounts for the distortion of objects seen through spectacles.

150. *To estimate the distortion of a distant object viewed by reflexion at a concave or convex mirror.*

Let E be the position of the eye, C the centre of the surface, so that EC is the axis of visual reference.



Let QRE be the axis of the small oblique pencil by which any point Q of a distant object is seen by E, q the position of the circle of least confusion of this pencil,

$$AC = r,\quad AE = x,\quad AR = y;\ \therefore\ AM = \frac{y^2}{2r}.$$

Then considering the rays from Q to be parallel, we have

$$v_1 = \frac{r\cos\phi}{2},\quad v_2 = \frac{r}{2\cos\phi},\quad \text{and } Rq = \Delta, \text{ where}$$

$$\frac{1}{\Delta} = \frac{v_1\cos\phi + v_2}{v_1 v_2 (1+\cos\phi)} = \frac{\dfrac{\cos\phi}{v_2} + \dfrac{1}{v_1}}{1+\cos\phi} \quad \text{(Art. 70)},$$

$$= \frac{2}{r}\left\{\frac{\cos^2\phi + \sec\phi}{1+\cos\phi}\right\} = \frac{2}{r}\left(\frac{1-\phi^2+1+\dfrac{\phi^2}{2}}{1+1-\dfrac{\phi^2}{2}}\right) = \frac{2}{r},$$

if ϕ &c. be neglected;

$$\therefore\ \Delta = Rq = \frac{r}{2}.$$

Also a ray diverging from Y would after reflexion pass through E;

$$\therefore\ \frac{1}{AY} + \frac{1}{x} = \frac{2}{r} + \left(\frac{1}{r} - \frac{1}{x}\right)^2 \frac{y^2}{r} \quad \text{(Art. 52)};$$

$$\therefore\ \frac{x}{AY} = \frac{2x}{r} - 1 + \left(\frac{1}{r} - \frac{1}{x}\right)^2 \frac{y^2 x}{r}$$

$$= \left(\frac{2x}{r} - 1\right)\left\{1 + \frac{\left(\dfrac{1}{r} - \dfrac{1}{x}\right)^2}{\dfrac{2}{r} - \dfrac{1}{x}} \cdot \frac{y^2}{r}\right\} \quad \text{..........(i)};$$

$$\therefore \frac{\tan\epsilon}{\tan\eta} = \frac{EM}{YM} = \frac{x - \frac{y^2}{2r}}{AY - \frac{y^2}{2r}} = \frac{x}{AY}\left\{1 + \left(\frac{1}{AY} - \frac{1}{x}\right) \cdot \frac{y^2}{2r}\right\}$$

$$\text{becomes} = \left(\frac{2x}{r} - 1\right)\left\{1 + \frac{\left(\frac{1}{r} - \frac{1}{x}\right)^2}{\frac{2}{r} - \frac{1}{x}} \frac{y^2}{r}\right\}\left\{1 + \left(\frac{1}{r} - \frac{1}{x}\right)\frac{y^2}{r}\right\},$$

by writing for $\frac{x}{Ay}$ its value from (i) and

putting $\quad \frac{1}{AY} = \frac{2}{r} - \frac{1}{x} \quad$ in the small term

$$= \left(\frac{2x}{r} - 1\right)\left[1 + \left\{\left(\frac{1}{r} - \frac{1}{x}\right) + \frac{\left(\frac{1}{r} - \frac{1}{x}\right)^2}{\frac{2}{r} - \frac{1}{x}}\right\}\frac{y^2}{r}\right]$$

$$= \left(\frac{2x}{r} - 1\right)\left\{1 + \left(\frac{1}{r} - \frac{1}{x}\right)\frac{\frac{3}{r} - \frac{2}{x}}{\frac{2}{r} - \frac{1}{x}} \cdot \frac{y^2}{r}\right\};$$

$$\therefore \frac{\tan\eta}{\tan\epsilon} = \frac{r}{2x - r}\left\{1 - \frac{(x - r)(3x - 2r)}{2x - r} \cdot \frac{y^2}{r^2 x}\right\} \ldots\ldots\ldots\ldots \text{(ii)}.$$

Again, $Em = x - \frac{y^2}{2r} - Mm = x - \frac{y^2}{2r} - Rq \cdot \cos\eta$

$$= x - \frac{y^2}{2r} - \frac{r}{2}\left(1 - \frac{\eta^2}{2}\right) = x - \frac{y^2}{2r} - \frac{r}{2}\left(1 - \frac{y^2}{2x^2}\right)$$

$$= \frac{2x - r}{2}\left(1 - \frac{2x^2 - r^2}{2x - r} \cdot \frac{y^2}{2rx^2}\right) \ldots\ldots\ldots\ldots \text{(iii)}.$$

If then D be the distance of the object, we have

$$\frac{qm}{D\tan\epsilon}=\frac{Em}{D}\cdot\frac{\tan\eta}{\tan\epsilon}=\frac{r}{2D}\cdot\left\{1-\frac{2x^2-r^2}{2x-r}\cdot\frac{y^2}{2rx^2}-\frac{(x-r)(3x-2r)}{2x-r}\cdot\frac{y^2}{r^2x}\right.$$

$$\frac{r}{2D}\left(1-\frac{6x^3-8rx^2+4r^2x-r^3}{2x-r}\cdot\frac{y^2}{2r^2x^2}\right)$$

$$=\frac{r}{2D}(1+Ay^2), \text{ suppose.}$$

Now this ratio $\frac{qm}{D\tan\epsilon}$ represents the ratio of the distances from the axis of the mirror—of a point seen in the image, and of the corresponding point of the object. The term Ay^2 may be taken as measuring the distortion, and the algebraic sign of A will indicate the nature of the deformation of the image.

Thus for example, suppose the object presents a series of horizontal and vertical parallel lines, fig. 1, Art. 149,—as a bookcase in a distant part of the room—the distortion becomes greater for greater values of y, i.e. for points more and more remote from the axis of the mirror, and is of the character presented in fig. 2 or fig. 3, according as A is positive or negative.

(i) For a concave mirror, if x be sufficiently great,—as for instance $x > r$,—A is negative, and the distortion is of the character of fig. 3: the image being inverted.

(ii) For a convex mirror, r is negative and A negative also for all values of x, and the distortion in this case also is as in fig. 3—the image being erect.

But in this case the numerical value of A will be greater than in the case of a concave mirror, and therefore the distortion is more marked and disagreeable.

151. *Brightness of an Image.*

Consider the image formed by a lens of a plane object of small but sensible magnitude: let d, d' be the distances from the lens of the object and image, Δ the diameter or aperture of the lens: I the intrinsic illuminating power of the object.

Now the amount of illumination emanating from the object and intercepted by the lens may be measured by

$$\text{area object} \,.\, I \,.\, \frac{\pi\Delta^2}{4d^2},$$

and this is diffused over the image;

∴ illumination at a point of the image

$$= \frac{I\pi\Delta^2}{4d^2} \cdot \frac{\text{area of object}}{\text{area of image}}, \quad \text{but} \quad \frac{\text{area of object}}{\text{area of image}} = \frac{d^2}{d'^2};$$

∴ illumination at a point of the image $= \frac{I\pi}{4}\left(\frac{\Delta}{d'}\right)^2$, which $\propto$ apparent magnitude of lens (*in area*) as seen from the image.

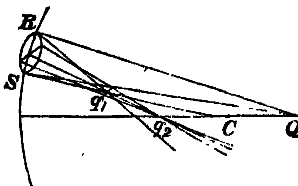
A similar result will follow for the brightness of an image formed by reflexion at a spherical mirror, and is, as we see, independent of the distance of the object.

Thus the illumination of an image formed by a lens or mirror,—(supposing no light lost by reflexion or refraction or absorption by the screen which receives the image)—is the same as would be produced by the direct light of the lens or mirror if it were equally luminous with the surface of the object which emits the light.

This supposes the object to be of sensible magnitude. When the object and its image are physical points—as a star and its image—the eye judges only of absolute light, and the brightness of the image is proportional to the density of rays concentrated in it, i.e. the brightness and apparent magnitude of the lens as seen from the object. Thus in the case of a star whose distance is constant, the absolute brightness of the image is proportional to the area of the object-glass: hence the importance of large telescopes in observing very faint stars.

152. *To measure the comparative density of the rays at different points of a catacaustic.*

Let Q be the radiant point, QC the axis of the mirror, QR the axis of a small pencil QRS, of which q_1, q_2 are the primary and secondary foci after reflexion, ϕ the angle of incidence, I the intrinsic intensity of rays diverging from Q.



Then in the primary plane, condensation of rays at q_1

$$\propto \frac{Rq_1}{RQ},$$

which is obtained by comparing the breadths of sections of the reflected and incident pencils taken at small equal distances from q_1 and Q—and assuming the condensation to vary inversely as the breadth;

And in the secondary plane the condensation at q_1 (and all along the primary focal line)

$$\propto \frac{1}{QR} \cdot \frac{Rq_2}{q_1q_2},$$

therefore on the whole, condensation at $q_1 \propto$ product of these two

$$\propto \frac{Rq_1 \cdot Rq_2}{QR^2 \cdot q_1q_2}.$$

The above neglects the rays which may pass through q_1 after reflexion at other points of the mirror, which however would vanish compared with the condensation of rays in the primary plane.

The expression becomes indefinitely large at the geometrical focus, since q_1q_2 is there $= 0$—that is, the condensation of rays at the geometrical focus is very much greater than at other points of the caustic.

A similar measure might be obtained in the case of a diacaustic,—but such measures cannot be expressed very simply in analytical terms.

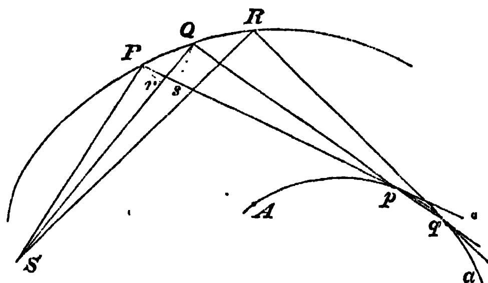
Note. Some cases of the more general problem of determining the law of condensation of light at any point in a system of rays proceeding according to any law, and the law of illumination of a surface on which the system is received—are discussed in a paper *On Systems of Rays* in the *Messenger of Mathematics*, Vol. III. p. 33, by Mr H. J. Sharpe.

153. *Length of the curve of a caustic by reflexion or refraction.*

(i) By reflexion.

Let S be the radiant point, SP, SQ, SR three consecutive rays, the first two intersecting in p after reflexion, the last two intersecting in q. p, q are ultimately points on the caustic, and pq is ultimately a small arc of the caustic.

Let $SP = \rho$, $Pp = \rho'$, $Ap = \sigma =$ length of caustic measured



from some fixed point A on it. Draw Pr, Qs, perpendiculars from P on SQ and from Q on Pp.

Then if ϕ be the angle of incidence at P, we have

$$Qr = PQ \sin \phi = Ps, \text{ nearly},$$

and therefore if $\Delta\rho$ be the small change of ρ in passing from P to Q,

$$\Delta\rho = Qr = Ps = Pp - Qp = Pp - (Qq - pq)$$

$$= \rho' - (\rho' + \Delta\rho') + \Delta\sigma;$$

$$\therefore \Delta\sigma = \Delta\rho + \Delta\rho'.$$

This which is approximately true becomes strictly true in the limit; therefore integrating

$$\sigma = \rho + \rho' + \text{a constant},$$

ρ, ρ' are supposed to be measured positive in the direction in which light is propagated.

Ex. The whole length of the caustic in the example of *Art.* 71,

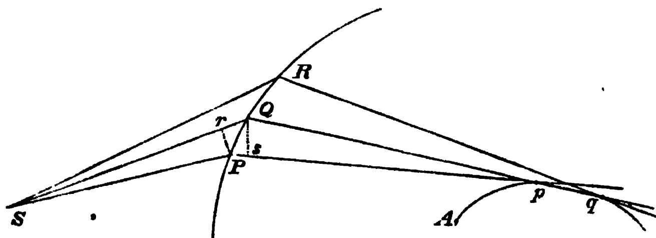
is $=$ *radius of mirror:*

and in the example of *Art.* 72, the whole length of the caustic

is $= \frac{16}{3}$ *radius of mirror.*

(ii) By refraction.

With notation similar to the preceding,



μ being the index of refraction from first medium into the second, we have

$$\begin{aligned}
\Delta\rho = Qr = PQ \,.\sin\phi &= \mu PQ \sin\phi' \\
&= \mu Ps = \mu\,(Pp - Qp) \\
&= \mu\,\{Pp - (Qq - pq)\} \\
&= \mu\,\{\rho' - (\rho' + \Delta\rho') + \Delta\sigma\};
\end{aligned}$$

$$\therefore \Delta\sigma = \frac{\Delta\rho}{\mu} + \Delta\rho';$$

$$\therefore \sigma = \frac{\rho}{\mu} + \rho' + \text{a constant}.$$

Thus we see that the length of any catacaustic or diacaustic corresponding to any pencil in one plane can be expressed in algebraic terms.

In each of the above cases (i), (ii), ρ, ρ' are supposed to be measured positive in the direction in which light is propagated.

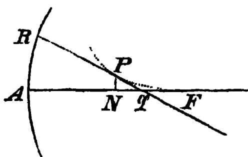
Analytical investigations of the above results are given in Herschel's *Light* and other works.

COR. When two consecutive reflected *or refracted* rays are parallel, it indicates an asymptote in the caustic. If ϕ be any variable which defines the direction of an incident ray, ψ the angle which the corresponding reflected *or refracted* ray makes with a fixed direction,—then for an asymptote in the caustic we must have $\frac{d\psi}{d\phi} = 0$.

If the value of $\frac{d\psi}{d\phi}$ does not vanish for any admissible value of ϕ, there is no asymptote.

154. *The form of a caustic near a cusp is a semicubical parabola.*

The tangent at a cusp is perpendicular to the reflecting or refracting surface, and the aberration is proportional to the square of the breadth of a pencil, when it is small. Take then F the cusp as origin, and Fq the tangent at F to the caustic as axis of x—this corresponds to the axis of a direct pencil after reflexion or refraction.



$FN = X$, $PN = Y$, co-ordinates of any point in the caustic, then

$$X - Y\frac{dX}{dY} = Fq = \text{aberration of } Rq \propto RA^2,$$

$$\propto Aq^2 . \tan^2 RqA \propto \tan^2 RqA, \text{ nearly,}$$

$$= a \cdot \left(\frac{dY}{dX}\right)^2, \text{ suppose,}$$

for a part of the caustic not far from the cusp.

Transforming we have

$$Y = X\frac{dY}{dX} - a\left(\frac{dY}{dX}\right)^3 \text{.....................(i),}$$

which is a differential equation of Clairaut's form.

The complete solution is

$$Y = CX - aC^3 \text{.....................(ii),}$$

which in fact by varying C gives the successive ray-tangents to the caustic. The envelope of these, or the singular solution of (i), gives the caustic.

Differentiate (ii) with respect to C, and we get

$$X - 3aC^2 = 0 \text{ ....(iii).}$$

Eliminate C between (ii) and (iii), and we have

$$aY^2 = \frac{4}{27} \cdot X^3,$$

for the form of the caustic near the cusp.

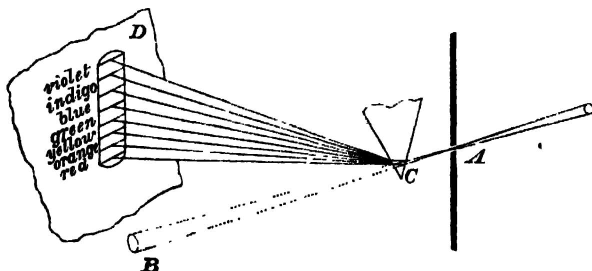
CHAPTER IX.

ON THE CHROMATIC DISPERSION OF LIGHT.

155. In the previous chapters pencils of light have been considered homogeneous. The following experiment of Newton shews that a pencil of sunlight has not this uniform nature, but admits of decomposition into a system of pencils, in each of which the rays have a peculiar degree of refrangibility.

156. *Newton's Experiment.*

If the light of the sun be admitted into a darkened room through a small aperture A in a window-shutter, the pencil



of light after entering the room may be regarded as approximately a cone, with A for its vertex, and the sun's apparent diameter for its vertical angle. If this pencil be allowed to fall perpendicularly on a screen, a circular bright spot B of white light is visible. Let the pencil now be refracted upwards through a prism of glass, *or any other refracting substance*, placed very near to the aperture A, the axis of the pencil passing perpendicularly to the edge of the prism C, which for convenience of reference we will suppose horizontal, and very near to it.

If the pencil be now received perpendicularly on a screen, an elongated stripe, *or spectrum* D is seen. On turning the prism continuously about its edge, this spectrum will descend to a certain limiting position, and then ascend. It will be in this limiting position when the prism is in a position of *minimum deviation*, which is indicated by the spectrum remaining stationary for a *very small* angular motion of the prism about its edge in *either* direction. If the spectrum be examined in this position, the distance CD being equal to CB, it is found to be of the same horizontal breadth as the circular spot B, but about five times longer, and of different colours in different parts, being *red* at the lowest, or least refracted end, then by a gradual change of tint becoming *orange, yellow, green, blue, indigo, violet* in succession, as we proceed to the upper extremity.

Now, since the axis of the pencil passes with minimum deviation in a principal plane of the prism, and very near its edge,—if light were homogeneous the refracted pencil would be a cone diverging from an origin at a distance $= CA$ from C, and having the sun's apparent diameter for its vertical angle (Art. 96, Cor. 2). This cone being received perpendicularly on a screen at a distance $CD = CB$ the appearance would be a circular spot exactly equal to B. The experiment therefore leads to the following theory:

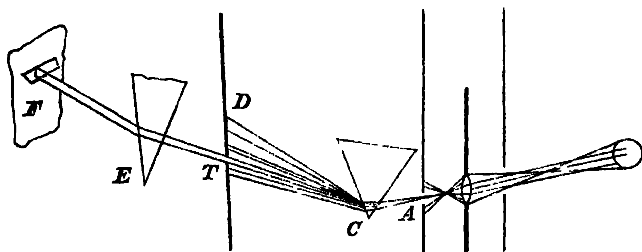
Sun-light consists of different species of light of all degrees of refrangibility within certain limits, and of all varieties of colour. The red rays are the least, and the violet rays the most refrangible.

Obs. The elongation of the spectrum in this experiment, as stated above, is that which was observed by Newton with the prism used by him. Its amount in different cases depends on the refracting angle of the prism and its material.

If the whole length of the spectrum (AI in the figure, p. 152) be divided into 360 equal parts—the relative lengths of the spaces occupied by the different colours may be taken to be *red*, 56; *orange*, 27; *yellow*, 27; *green*, 46; *blue*, 48; *indigo*, 47; *violet*, 109,—a flint-glass prism being supposed to be employed, but the proportion will vary considerably with prisms of different materials.

157. The spectrum formed in the manner just described will not be *pure*, that is, the coloured light at any point of it will consist of a mixture of rays of a higher degree of refrangibility from one point of the sun's disc with rays of a lower degree of refrangibility from a lower point; and consequently in this spectrum the rays of different degrees of refrangibility are not accurately separated. The defect arising from this over-lapping of the spectra which correspond to pencils emanating from successive points of the sun's disc, may be diminished by receiving the pencil which enters at A (fig. p. 148) on a screen perforated so as to allow only a small part of this pencil to pass, and thus the pencil incident upon the prism would have a smaller vertical angle, and diverge more nearly from a point.

But a better plan is to obtain a *very small* image of the sun by means of a lens of small focal length (as in the figure),



and a small part of the pencil diverging from this image may be allowed to pass through the aperture A and received on the prism.

If, for example, the focal length of the lens be 1 inch, the diameter of the sun's image formed at its principal focus = tan (sun's apparent diameter) = tan 30′ nearly = $\frac{1}{114}$ inch, which is so small a quantity that for this purpose it may be regarded as a physical point.

If the screen D be moveable, and perforated at T, any part of the spectrum first formed can be *insulated* and separately examined *or analyzed* by a second prism E.

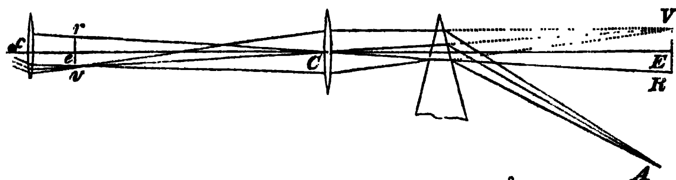
By this means a high degree of purity in the spectrum may be obtained; when this is done, it is found that the colours admit of no further decomposition, since when analyzed by a second prism at *E*, and received on a screen *F*, the appearance there is a spot of uniform colour, the same as at *T*, and of dimension the same as if the pencil at *T* had been allowed to pass uninterrupted to the screen *F*.

158. Instead of a very small aperture (*A*) Wollaston and Fraunhofer admitted the sun's light through a very narrow slit at a considerable distance from the prism and parallel to its edge. Thus a spectrum of considerable *breadth* was obtained without affecting its purity,—the effect of the slit being to give an assemblage of innumerable linear spectra placed side by side, the appearance resembling a striped riband, with colours arranged in direction of its breadth and parallel to the edge of the prism.

Wollaston observed this spectrum with the naked eye placed close to the prism. Fraunhofer employed a telescope.

An apparatus constructed and arranged for observing the spectrum is called a *spectroscope*. Very elaborate and costly ones have been employed by Mr Gassiot, Mr Huggins and others: some very simple *direct-vision* spectroscopes are now constructed for ordinary purposes,—the arrangement of the prisms in the eye-piece being such as to produce *dispersion* without *deviation*, or to cancel the *deviation* by means of internal reflexions within the prisms.

159. Let *A* be a section of the slit made by a plane per-



pendicular to the edge of the prism, *C* the object-glass, *c* the eye-piece of the telescope. Then if the prism be in such

a position that the deviation is a minimum, all the rays of a given refrangibility after refraction through the prism will diverge from some point E at the same distance as A from the prism, and after refraction at the lens C will converge to e. Less refrangible pencils will diverge from points towards R, after refraction through the prism, and will converge to points towards r after refraction through the lens. In like manner, more refrangible light after being refracted through the prism and lens will converge to points towards v. In the spectrum thus obtained the light at any point consists of rays of a definite degree of refrangibility free from admixture with light of a different refrangibility. This *real* and *pure* spectrum can then be examined by a suitable eye-piece at c.

If the telescope be furnished with a micrometer, the position of any colour *or fixed line* of the spectrum may be accurately determined.

160. In observing the spectrum formed in this manner by the light of the sun, some remarkable facts were discovered by Fraunhofer (A.D. 1814), viz. that the *continuity of colour* in the spectrum was not complete, but that it was interrupted by nearly 600 *dark lines*—transverse to the length of the spectrum *or* parallel to the edge of the prism—the strongest of which subtended in breadth an angle of from 5″ to 10″. The positions of a few of the most remarkable of these lines are indicated in the figure.

A a B C *E b F* *G* *H*

violet

A is a well-defined line a little within the *red* end of the spectrum. At a a group of several lines form a band, B is a well-defined line and of sensible breadth. In the space between B and C there are 9 very fine lines, C is a very dark

line. Between *C* and *D* 30 very fine lines may be counted. At *D* in the *orange* are two strong lines separated by an extremely small interval. Between *D* and *E* about 84 lines may be distinguished,—*E* lies in the *green;* it consists of several lines of which the middle line is rather broader than the others,—so close that they appear to form one dark line. On both sides of *E* are other groups of fine lines much resembling *E*, but not quite so dark. Between *E* and *b* are about 24 lines, at *b* are 3 strong lines of which the two furthest from *E* are very close. These are the strongest lines in the bright part of the spectrum. Between *b* and *F* about 50 lines may be counted. *F* is a strong line at the commencement of the *blue.* Between *F* and *G* may be counted about 185 lines variously grouped and of various breadths. *G* is a strong line in the *indigo* in the midst of a band of very fine lines. Between *G* and *H* are about 190 lines of various sizes. *H* is a strong line in the violet in the middle of a band of fine lines. Near it, but further from *G*, a similar band is seen. From *H* to *I*, the end of the spectrum, the lines are equally numerous.

Two of the fixed lines, probably *E* and *F*, had been discovered by Wollaston previously to the experiments of Fraunhofer—but he did not pursue the investigation of them.

161. The name of *fixed lines* was given by Fraunhofer to these dark bands which interrupt the continuity of the colours. As long as the source of light remains the same, these lines occur in the *same order* and in the *same colours*, whatever be the substance of which the prism is formed,—their relative distances only varying when different substances are employed.

Thus for instance with solar light—whether obtained directly from the Sun, or indirectly, as from the clouds or sky; from the Moon or planets, or from a rainbow—the phenomena of the dark lines in the spectrum, as to order, number, and relative position, are always the same when the same substance is used.

When the Sun is very near the horizon the blue end of the spectrum disappears, and lines are seen in the remaining part which were not before visible.

Analogous effects are produced by interposing certain coloured vapours and liquids between the slit A and the source of light.

The spectrum formed by electrical light was observed by Professor Wheatstone to consist almost entirely of a few bright lines which varied with the substance between which the spark was produced.

161*. Certain substances when exposed to the rays of the Sun acquire the faculty, without being sensibly heated, of emitting during a limited time (which varies in length with different substances) a luminosity whose brightness is sensible in the dark. This phenomenon, which is called *phosphorescence*, is developed chiefly by the ultra-violet rays.

There are other substances—as *fluor-spar*, *sulphate of quinine*, *&c.*—which exhibit a similar phenomenon, called *fluorescence*, but enduring only so long as the sun's rays fall upon them.

These phenomena have been carefully studied by Professor Stokes and M. E. Becquerel; they seem to indicate that under certain circumstances the rays of light are capable of undergoing a change of refrangibility.

162. Fraunhofer selected the fixed lines B, C, D, E, F, G, H as points of reference in the spectrum, and by measuring (Art. 165) with extreme accuracy the deviations of these lines through prisms of the same substance with different refracting angles, he ascertained that for a ray corresponding to any one of the fixed lines, the ratio of the sine of the angle of incidence to the sine of the angle of refraction was invariable,—thus affording the strongest corroboration of the law of refraction (Art 9).

The following table contains some of the results of

	Sp. G.	μ_B	μ_C	μ_D	μ_E	μ_F	μ_G	μ_H
Water at 15° *R*	1·000	1·33095	1·33171	1·33357	1·33585	1·33780	1·34127	1·34417
Crown Glass, No. 9	2·535	1·52583	1·52685	1·52959	1·53301	1·53605	1·54166	1·54657
Flint Glass, No. 13	3·723	1·62775	1·62968	1·63504	1·64202	1·64826	1·66029	1·67106
Oil of Turpentine	0·885	1·47050	1·47153	1·47443	1·47835	1·48174	1·48820	1·49387

Fraunhofer's observations as given in his *Essay on the Determination of Refractive Powers, &c.*—the refractive index of a fixed line being understood to mean that of a ray which would suffer the same deviation if it existed.

Crown glass is made of fine white sand and kelp or pearl ashes—it is colourless and is the best kind of glass employed for glazing windows, plate glass, &c.

Flint glass or *crystal* is of a pale green hue and is made of silica, lead, and potash in proportions of about $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$—but varying in different specimens. The proportion of lead is increased to increase the refractive power which increases with the density of the medium. See table of refractive indices, p. 179.

162 (*a*). The analysis of light by means of the spectrum produced by a prism—what is commonly called *spectrum analysis*—has become a subject of great interest and careful observation since the important discoveries of Kirchhoff, who (A.D. 1859) propounded as a natural law *that if a vapour when sufficiently heated possesses the property of emitting light of certain refrangibilities, that vapour at a lower temperature has a tendency to absorb, or refuse a passage to, light of the same refrangibilities which may be incident upon it*:—and he demonstrated the law experimentally in the case of certain metals—*Sodium, Lithium, Potassium, &c.*

This law with respect to *light* is analogous to the *law of exchanges* with respect to *heat*, as laid down by Dr Balfour Stewart, which asserts that the relation between the amount of heat emitted and that which is absorbed at any *given temperature* remains *constant for all bodies*;—and that the greater the amount of heat emitted the greater must be the amount of heat absorbed. In order however that this law of exchanges as thus formulated with respect to heat-giving rays may hold good for luminous rays, Professor Roscoe insists that the emissive and absorptive powers of substances must be compared at the same temperature (*Spectrum Analysis*, p. 214. 2nd ed. 1870).

162 (β). The spectra of light emanating from different substances, when decomposed *or analyzed* by a prism, may differ in many important respects from each other. Fol-

lowing Mr Huggins, one of the most careful and accurate of experimenters, we may classify all the spectra which can present themselves in three general groups or orders.

(i) A spectrum of the *first order* is *continuous*, unbroken by *dark* or *bright* lines. Such a spectrum is afforded by light emitted from an *opaque* body in a state of incandescence—and almost certainly from matter in the *solid* or *liquid* state. A spectrum of this order gives us no information of the chemical nature of the body from which the light emanates.

(ii) A spectrum of the *second order* is *discontinuous*, consisting of colour *lines* of light, separated from each other. Such a spectrum is afforded by light emitted from luminous matter *in the state of gas*. Each element and every compound body that can become luminous in the gaseous state without suffering decomposition, is distinguished by a group of lines peculiar to itself. Thus the vapour of

Sodium gives for its spectrum the bright double-line *D*, in the *orange-yellow*.

Lithium gives a bright *red* line between *B* and *C*, and a weak *yellow* line between *C* and *D*.

Potassium gives a *red* line coinciding with *A*, and a pale *violet* line not far from *H*.

Thallium gives a bright *green* line not far from *E*.

Hydrogen gives three lines coincident with *C*, *F*, *G*, in the *red*, *bluish-green* and *violet*.

Nitrogen and *oxygen* give more numerous lines than *hydrogen*.

Iron gives a very complex system of lines—

and other substances give lines more or less numerous, and in most cases besides the few predominant lines there are faint traces of others.

(iii) In the *third order* of spectra the continuity of the coloured light is broken by *dark lines*. These dark spaces do not arise from the source of light, but from the absorption of definite colours—(or rates of vibration, if we are regarding the physical theory of light)—by *vapours* through which the light has passed. Such spectra are formed by the light of the sun and stars.

162 (γ). It was shewn by Kirchhoff that if vapours of terrestrial substances come between the eye and an incandescent body, they cause *groups of dark lines*—and that the group of *dark* lines produced by each vapour is identical in the number of lines, and in their position in the spectrum with the group of *bright* lines of which the light of the vapour consists when it is luminous.

By this discovery Kirchhoff interprets the dark lines in the solar spectrum:—he infers that of the light emitted from the incandescent body of the sun, rays of various refrangibilities are absorbed in passing through the glowing vapours of various substances present in the Sun's atmosphere: and he concludes with certainty that *iron, sodium, magnesium, hydrogen, calcium, barium, copper, zinc, &c.* are present in the Sun's atmosphere in a state of luminous gas:—but he finds no trace of *gold, silver, antimony, mercury, aluminium, tin, lead, lithium, &c.* See Roscoe, *Spectrum Analysis.* Also, Report upon the present state of our knowledge of Spectrum Analysis, presented to the British Association, 1880:—Also, Professor W. G. Adams, *Nature*, Vol. XXII., p. 411.

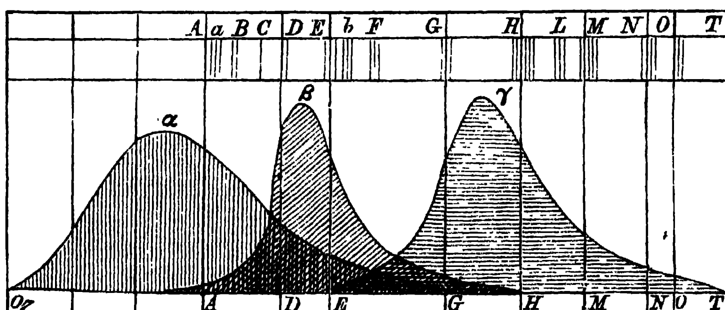
162 (δ). It is obvious that if a comparison be made of the spectrum of the light emanating from an unknown source (as the Sun, a star, or a nebula) with the spectra of different terrestrial elements, as standard spectra, important information may be obtained as to their chemical constitution. The spectra of several fixed stars have been observed with great care by Mr Huggins and Dr W. A. Miller, and consist in most cases of very numerous bands of dark lines, a comparison of which with the solar spectrum shews that many terrestrial elements which are present in the Sun are also present in the several stars: that some which are present in the Sun are not present in particular stars, and *vice versâ*—for example, hydrogen is present in the Sun's atmosphere and in that of *Aldebaran*, but is wanting in that of *Betelgeux.*

Mr Huggins has also examined the spectra of several nebulæ, and the results are very remarkable. The spectra of some of them are of the *second order*, indicating that the matter of which they are composed is in a gaseous state. For full information, however, on the various branches of

spectrum analysis the student may consult Roscoe's and Huggins' *Spectrum Analysis.*

162 (ϵ). If a pencil of sun-light be analyzed by a prism and examined with respect to the *thermal* or *heat-giving* power, and the *chemical* or *actinic* power, as well as the *luminous* power, it is found that what may be called the *thermal* spectrum, the *colour* spectrum, and the *chemical* spectrum, overlap each other:—the *thermal* spectrum extending considerably beyond the *red* end of the *colour* spectrum, and the *chemical* spectrum considerably beyond the *violet.*

In the following diagram the relative extension of the three several spectra are represented by the horizontal line



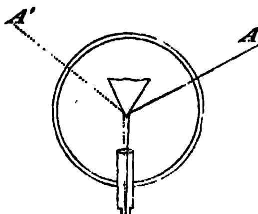
$O_r AB \ldots OT$:—the letters $A, B, C \ldots H$ corresponding to the several *lines* in the colour spectrum (Art. 160):—and the ordinates bounded by the curves α, β, γ, representing the intensities at the corresponding points of the thermal spectrum, the colour spectrum, and the chemical spectrum severally.

163. The following two propositions will indicate the nature of the observations requisite for determining the index of refraction of a ray of any colour—the ray being defined by its position relative to the fixed lines of the spectrum.

164. *To measure the refracting angle of a prism.*

Let the prism be firmly fixed to a graduated circle provided with verniers,—the edge of the prism being at the

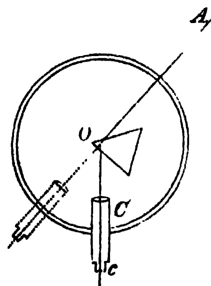
centre of the circle, perpendicular to its plane, and turned towards the object-glass of a telescope fixed to the circle. Let the circle be turned until the image of a well-defined distant object A, seen by reflexion at one of the faces of the prism, and viewed through the telescope, coincides with the intersection of the cross-wires at the principal focus of the object-glass of the telescope—and read off the verniers attached to the circle.



Turn the circle about its centre till the image of the same object seen by reflexion at the other face of the prism coincides with the intersection of the cross-wires—the effect will be the same as if we supposed the circle to be stationary, and the object A to revolve about the centre of the circle into a position A' such that it is seen by reflexion at the second face of the prism, and read off the verniers again. The *difference of the readings*—or the angle through which the circle has been turned—is equal to twice the angle of the prism.

165. *To measure the minimum deviation of a ray corresponding to one of the fixed lines, out of air into any medium formed into a prism; and thence to determine its index of refraction.*

Let O be the centre of a graduated circle moveable round an axis perpendicular to its own plane on a fixed circle carrying verniers, Cc a telescope the axis of the object-glass of which passes through the axis of the circle and is parallel to the plane of the circle,—A the intersection of the plane described by the axis of the telescope with the slit perpendicular to the plane of the circle, by which light is admitted. The telescope is provided with cross-wires at such a distance from the object-glass that when it is pointed to A, the image of A may be at the intersection of the wires. Place the prism with its edge perpen-



dicular to the plane of the circle, so that its faces may be equidistant from the axis of the circle. Turn the prism until the incident ray and the given emergent ray make equal angles with the faces of the prism—*i.e.* until the deviation of this particular ray is a minimum, or the image of the fixed line stationary. The prism retaining this position, turn the telescope till the image of the line coincides with the intersection of the wires and read the verniers.

Now remove the prism and turn the telescope until the intersection of the wires coincides with the image of A and read the verniers again. The difference of the two readings—or the angle through which the circle has been turned between the two observations—is the *minimum deviation* of the given ray.

If D be this deviation, ι the refracting angle of the prism, μ the index of refraction for the given colour, or fixed line, then with the notation of Art. (95),

$$D = 2\phi - \iota,\ \iota = 2\phi'; \quad \therefore \mu = \frac{\sin\phi}{\sin\phi'} = \frac{\sin\dfrac{D+\iota}{2}}{\sin\dfrac{\iota}{2}},$$

from which μ can be calculated numerically.

166. *Obs.* The refractive index of a fluid or a gas for a given fixed line may be found in the same manner by enclosing the fluid or gas in a hollow prism of glass, the sides of which are *plates* with their surfaces accurately parallel. The deviation of the axis of the pencil then arises entirely from refraction through the fluid.

To obtain the index of refraction from air into *vacuum*, the hollow prism must be *exhausted:* the deviation in this case will be negative, and the value of $\mu < 1$. Of course the refractive index from vacuum into air is the reciprocal of the value of μ thus found: the values of the *absolute* refractive indices can then be calculated (Art. 84).

From *vacuum* into *common air*, $\mu = 1{\cdot}000294$ nearly;—the density of the air at the earth's surface being to that of water as ·0013 : 1, nearly.

Note. When the prism is not sufficiently perfect to exhibit the fixed lines, we must select by estimation the particular part of the spectrum for which the index of refraction is required,—and measure the deviation as above.

As the spectrum produced by a given species of light—sun-light for instance—refracted through a given substance always presents the same phenomena, we may regard it as a *standard scale* of colours which can be reproduced at any time, and in which any particular tint may be defined and identified by reference to the fixed lines.

Thus the value of μ for any particular medium or substance is not invariable, but is susceptible of every possible value between certain limits: but rays corresponding to each particular value of μ will conform to the laws of reflexion and refraction which have been discussed in the previous chapters on the supposition of light being homogeneous.

167. *Def.* A ray of white light being decomposed by refraction at any surface into a beam of coloured rays, the angle between any coloured ray,—or ray corresponding to a fixed line,—and the original white ray produced, is the *deviation* of that colour or fixed line.

The difference of the deviations of two colours, or fixed lines, is the *dispersion* of those colours or fixed lines.

A discussion of the values of μ obtained in Fraunhofer's manner leads to the conclusion,—that the *ratio* of the dispersion of any two colours to the dispersion of the extreme colours of the spectrum, is not constant when different media are used. This fact is called the *Irrationality of dispersion.*

Hence if two prisms be formed of different substances and of such refracting angles as to give spectra of the same total length, i.e. equal total dispersion,—and they be placed so as to refract a pencil of white light in opposite directions,—in the emergent beam the extreme red and violet will be united but the intermediate colours will not be completely so, but will give rise to a spectrum of faint colours and small

breadth. Such spectra which exist in consequence of the irrationality of dispersion are called *secondary spectra.*

168. *When a ray of white light is refracted through a prism in a principal plane, to find the dispersion of two colours of given refractive index.*

Let ι be the refracting angle of the prism, ϕ the angle of incidence of the white ray, ϕ' the angle of refraction of the coloured ray for which μ is the refractive index,—and ψ', ψ angles of incidence and emergence of the same at the second surface,—D the deviation of this colour. Then

$$\sin\phi = \mu\sin\phi', \quad \sin\psi = \mu\sin\psi';$$

$$D = \phi + \psi - \iota, \quad \iota = \phi' + \psi' \quad \text{(Art. 92, } Obs. \text{ 2).}$$

If $\mu + \delta\mu$ be the index of refraction for another of the colours into which the incident ray is separated by refraction, $\delta\mu$ being small—$D + \delta D$ the deviation of this second colour—then will δD be the dispersion required.

Treating μ as an independent variable, since it is susceptible of all values between the limits corresponding to the extreme colours, ϕ being constant,

$$\delta D = \frac{d\psi}{d\mu}\,.\,\delta\mu, \quad 0 = \frac{d\phi'}{d\mu} + \frac{d\psi'}{d\mu},$$

$$\cos\psi\,\frac{d\psi}{d\mu} = \sin\psi' + \mu\cos\psi'\,\frac{d\psi'}{d\mu},$$

$$0 = \sin\phi' + \mu\cos\phi'\,\frac{d\phi'}{d\mu};$$

$$\therefore \frac{\cos\psi}{\cos\psi'}\,.\,\frac{d\psi}{d\mu} = \frac{\sin\psi'}{\cos\psi'} + \frac{\sin\phi'}{\cos\phi'} = \frac{\sin\iota}{\cos\phi'\cos\psi'}$$

therefore the dispersion

$$= \delta D = \frac{d\psi}{d\mu}\,.\,\delta\mu = \frac{\sin\iota}{\cos\psi\cos\phi'}\,.\,\delta\mu.$$

169. Cor. The above expression for the dispersion cannot vanish for any position of the prism. It admits however of a minimum value, which will happen when

$$\cos\psi\cos\phi' = maximum.$$

When such is the case, μ being constant in this problem,

$$\tan\psi \,.\, d\psi + \tan\phi' \,.\, d\phi' = 0;$$

but $d\phi' = -d\psi'$, and from $\sin\psi = \mu\sin\psi'$, we have

$$\cot\psi \,.\, d\psi = \cot\psi' \,.\, d\psi',$$

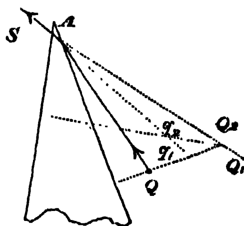
whence $$\tan^2\psi = \tan\phi' \tan\psi',$$

a relation which must obtain when the dispersion of two colours is a minimum; this combined with

$$\sin\phi = \mu\sin\phi', \quad \phi' + \psi' = i, \quad \sin\psi = \mu\sin\psi',$$

is sufficient to determine the direction of the incident ray when the dispersion is a minimum.

170. *To determine the position of any part of the spectrum seen through a prism.*



Let Q be the origin of a small pencil whose axis is obliquely refracted through a prism in direction QAS in a principal plane of the prism.

Let q_1 be the primary focus of the emergent pencil; then to an eye in AS the given rays will appear to diverge from a line passing through q_1 parallel to the original slit.

And with notation of (Art. 92, *Obs.* 2),

$$\sin\phi = \mu\sin\phi', \quad \sin\psi = \mu\sin\psi', \quad \phi' + \psi' = i,$$

$$Aq_1 = \frac{\cos^2\phi'\cos^2\psi}{\cos^2\phi\cos^2\psi'} \,.\, AQ,$$

from which equations the place of q_1 may be determined.

COR. Putting the result in the form

$$Aq_1 = \frac{(\mu^2 - 1)\tan^2\phi + \mu^2}{(\mu^2 - 1)\tan^2\psi + \mu^2} \cdot AQ,$$

it is obvious that the distance of q_1 from the prism is $> = <$ that of Q according as ϕ is $> = < \psi$.

171. *Def.* If μ_r, μ_v, μ be the indices of refraction for the extreme red and violet rays capable of producing a sensible impression on the eye of the observer, and for rays of mean refrangibility out of air into any medium, the quantity

$$\frac{\mu_v - \mu_r}{\mu - 1}$$

is called the *dispersive power* of the medium. This quantity is frequently denoted by the letter ϖ.

If the medium be formed into a prism with a small refracting angle i, and if D_r D_v D be the deviations for the extreme red and violet rays and for rays of mean refrangibility of the axis of a pencil which passes through the prism in a principal plane and at a small angle of incidence and emergence, then

$$D_r = (\mu_r - 1)\, i, \quad \text{(Art. 92, Cor.)}$$

$$D_v = (\mu_v - 1)\, i,$$

$$D = (\mu - 1)\, i;$$

$$\therefore \frac{D_v - D_r}{D} = \frac{\mu_v - \mu_r}{\mu - 1};$$

or the dispersive power of a medium formed into a thin prism is equal to the ratio of the total *dispersion* of the axes of the extreme red and violet pencils to the *deviation* of the axis of the pencil of light of mean refrangibility.

For an account of dispersive powers of second and higher orders to which the irrationality of dispersion gives rise, the student is referred to Herschel's *Treatise on Light*, (Art. 438).

Achromatic Combinations.

172. A combination of prisms or lenses is said to be *achromatic* when the *dispersion* of the pencils of light re-

fracted through them is reduced within the narrowest possible limits. We proceed to consider the conditions which must be satisfied in such combinations.

A pencil of sunlight after refraction does not in general converge to or diverge from a point, for two reasons; (i) from the unequal refrangibility of the different species of light of which it is composed, and (ii) in consequence of the finite breadth of the pencil and the curvature of the refracting surface. These causes of aberration being independent of one another may be separately considered. It will therefore be supposed for simplicity in the following articles, in investigating the conditions of achromatism, that no *spherical aberration*—(i.e. aberration arising from the *curvature* of the refracting surfaces,)—exists in the pencils which we consider. Moreover we shall consider it sufficient to obtain the conditions of achromatism *for the axis of a pencil*, since if these conditions are satisfied for the axis, they will be so very approximately for the other rays of the pencil,—supposing it small.

When we speak of *a colour* it must be understood to be defined by its position among the *fixed lines* of the spectrum —or to correspond itself to *a fixed line.*

173. The *possibility* of an achromatic combination arises from the fact that the *dispersion* of a ray and the *deviation* of any particular colour—or the *mean deviation,* if for convenience we take this as a definite measure of the deviation of the ray—produced by a refracting medium are not proportional. A comparison of the results given in the table (page 154) will prove this.

If *dispersion* were proportional to *deviation* for different media, then any combination which would destroy dispersion, would also destroy deviation; and in consequence would be useless for the purposes for which such combinations are designed. For instance, a combination of lenses in a telescope would have no magnifying power, if it did not produce deviation in the axis of a pencil transmitted through the telescope; but since in different media dispersion is not proportional to deviation, media can be found which produce the *same dispersion* in *opposite* directions of a given colour relatively to an-

other,—but at the same time, ***different deviations*** in ***opposite*** directions in the axis of a pencil. If a pencil then be refracted through these media, the two colours in question can be united, while the axis of the pencil suffers a deviation equal to the *difference*—or *algebraically, the sum*—of the deviations which the media would separately produce.

If *irrationality of dispersion* (Art. 167) had no existence, then in providing a combination such that two given colours should not be separated, we should simultaneously unite lights of all species. But since the colours are disproportionately dispersed in different media the other colours will in such a case be very nearly but not exactly united. A pencil therefore refracted through an achromatic combination will illuminate a screen with light still slightly coloured and give rise—as we have before stated—to *a secondary spectrum* (Art. 167). The fixed lines in this spectrum do not generally preserve the same order of succession which they have in the *primary spectra.* A combination of different media achromatic for all kinds of light being thus in general unattainable, it is customary to unite rays which are powerfully illuminating and also differ much in colour—the rest remaining but partially united.

174. *Achromatic Prisms.*

A pencil of light passes through two prisms of small refracting angles—passing in a principal plane of each—to find the condition of achromatism.

Let ι, ι' be the refracting angles of the prisms; μ, μ' the indices of refraction for a given colour (A). Then the deviations which the prisms would separately produce for this colour are $(\mu-1)\,\iota$ and $(\mu'-1)\,\iota'$. (Art. 92. *Obs.* 2.)

Hence the total deviation for this colour

$$=(\mu-1)\,\iota+(\mu'-1)\,\iota'.$$

If $\mu+\delta\mu$, $\mu'+\delta\mu'$ be the indices of refraction for another colour (B), then the total deviation for this colour will be

$$=(\mu+\delta\mu-1)\,\iota+(\mu'+\delta\mu'-1)\,\iota'.$$

If then the deviations be the same for the two colours A and B—i.e. if the combination be such as to unite these two colours,—we must have

$$0 = \delta\mu \,.\, \iota + \delta\mu' \,.\, \iota' \quad \text{.................(i)},$$

the condition required.

The ratio of ι to ι' being given by this equation (i) for any proposed pair of values of $\delta\mu$, $\delta\mu'$—that is for any two given colours, we can determine the values of ι and ι' if another relation between these be given,—suppose, for example, the combination is required to produce a given amount of deviation in a given colour, then we must have

$$(\mu - 1)\,\iota + (\mu' - 1)\,\iota' = \text{given deviation.........(ii)}.$$

And then (i), (ii) are sufficient to determine ι and ι'.

Note. The two colours A and B only will in general be united, since in consequence of irrationality, the ratio of $d\mu$ to $d\mu'$ is in general different for each pair of colours.

From equation (i) it is seen that the refracting angles ι, ι' have contrary signs—which indicates that the edges of the prisms must be turned towards opposite parts.

COR. Similarly if there were three prisms the condition that two colours A and B should be united after refraction through them, will be

$$0 = d\mu \,.\, \iota + d\mu' \,.\, \iota' + d\mu'' \,.\, \iota'' \quad \text{............(iii)},$$

where ι, ι', ι'' are the refracting angles of the prisms, and $d\mu$, $d\mu'$, $d\mu''$ the difference of the indices of refraction for A and B in the substances of which the prisms are severally composed. If $d\mu_1$, $d\mu_1'$, $d\mu_1''$ be similar quantities for A and a third colour C, then A and C will be united if

$$0 = d\mu_1 \,.\, \iota + d\mu_1' \,.\, \iota' + d\mu_1'' \,.\, \iota'' \quad \text{.............(iv)}.$$

If the conditions (iii), (iv) co-exist, then the three colours A, B, C would be united,—and a third condition, like that

of (ii) in the present Article, will render the values of ι, ι', ι'' perfectly determinate.

Similarly, if there were n prisms disposable, we might so determine their refracting angles as to unite n colours.

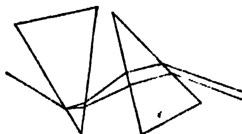
175. The following is a more general case of achromatism with two prisms.

A pencil of light passes through two prisms—the axis of the pencil passing in a principal plane of each: to find the condition of achromatism.

Let ι be the refracting angle of the first prism, μ its refractive index for a given colour, ϕ, ϕ' the angles of incidence and refraction at the first surface of this first prism, and ψ, ψ' the angles of emergence and incidence on the second surface of the axis of a pencil of light corresponding to this colour;

$$\therefore \sin\phi = \mu \sin\phi', \quad \sin\psi = \mu\sin\psi', \quad \phi' + \psi' = \iota.$$

If $\mu + d\mu$ be the index of refraction for another colour, and $\phi' + d\phi'$, $\psi' + d\psi'$, $\psi + d\psi$ the values of ϕ', ψ', ψ for this colour, then as in Art. (168), we shall have



$$d\psi = \frac{\sin\iota}{\cos\phi'\cos\psi} \cdot d\mu.$$

Let the pencil now be refracted through a second prism, and let ι_1 be its refracting angle, ψ_1, $\psi_1 + d\psi_1$ the angles of incidence of the axes of pencils corresponding to the two colours above considered, ϕ_1' the angle of incidence on the second surface of the second prism of the axis of the pencil for which the index is μ_1, and let $\mu_1 + d\mu_1$ be the index for the second colour.

Then if the axes of the pencils of the two colours are parallel at emergence from the second prism,—(which requires the prisms to be placed so that the deviations produced by them are in opposite directions),

$$d\psi_1 = \frac{\sin\iota_1}{\cos\phi_1'\cos\psi_1} \cdot d\mu_1,$$

and if the axes of the two colours emerge parallel we must have $d\psi = d\psi_1$;

$$\therefore \frac{\sin \iota}{\cos \phi' \cos \psi} d\mu = \frac{\sin \iota_1}{\cos \phi_1' \cos \psi_1} d\mu_1,$$

the condition required.

If the prisms are of the same substance, then $d\mu = d\mu_1$.

If the deviation be a *minimum* in *each* prism

$$\tan \phi \frac{d\mu}{\mu} = \tan \phi_1 \frac{d\mu_1}{\mu_1}.$$

176. *Note.* Since only a limited number of colours can be united by an achromatic combination,—it will be best to select colours for this purpose which have a brilliant illuminating power and also are as nearly as may be *complementary* (see Art. 185)—since by this course a far greater concentration of light and a more approximate union of the remaining colours will be produced, than if we attempted to unite the extreme red and violet rays which are too little luminous to render their union a matter of importance.

For example, if *two* colours only are to be united, the best to be selected for this purpose are those defined by the fixed lines C and F, or D and F (Art. 160). If there be a sufficient number of disposable quantities to enable us to unite *three* colours, the best to be selected would be the fixed lines C, E, and G,—or C, F, and a colour midway between D and E.

177. *Achromatic Lenses.*

In forming an achromatic combination of lenses the problem is different according as the refracted pencil passes *centrically* or *excentrically*. In the former case the pencils of different colours into which an incident pencil is separated have a common axis (Art. 111) in which their points of divergence or convergence lie (Art. 113), and the combination is achromatised by making as many as possible of these points coincide. But in a pencil refracted excentrically, the axes of the coloured pencils after refraction have different directions, so that there is an angular separation of their points of divergence or convergence—as seen by an eye in a given

position—and the condition of achromatism will be that which will render the axes of as many as possible of these coloured pencils parallel—since rays which enter the eye in a state of parallelism affect the eye in the same way as if they were coincident—or very nearly so.

178. *A pencil of light passes centrically with small obliquity through two thin lenses in contact—to find the condition of achromatism.*

A centrical pencil whose obliquity is small converges to or diverges from a point at nearly the same distance from the centre of the lens as a direct pencil (Art. 113).

It is thus sufficient to find the condition of achromatism of a direct pencil refracted through two proposed lenses.

The relation between u, v the distances of the conjugate foci of the pencil from the lenses is

$$\frac{1}{v}-\frac{1}{u}=\frac{1}{f_1}+\frac{1}{f_2} \text{ (Art. 114)},$$

f_1 being the focal length of the first lens,—the radii of whose surfaces are r_1, s_1—for a colour *or fixed line* of the spectrum, whose refractive index is μ_1:—similar quantities for the other lens being denoted by the suffix 2.

Hence $$\frac{1}{f_1}=(\mu_1-1)\left(\frac{1}{r_1}-\frac{1}{s_1}\right);$$

therefore the alteration of $\frac{1}{f_1}$ for an alteration $d\mu_1$ of μ_1, is

$$d\mu_1\left(\frac{1}{r_1}-\frac{1}{s_1}\right), \text{ which is } =\frac{d\mu_1}{\mu_1-1}\cdot\frac{1}{f_1},$$

and the alteration of $\frac{1}{f_2}$ for an alteration $d\mu_2$ of μ_2, is

$$\frac{d\mu_2}{\mu_2-1}\cdot\frac{1}{f_2}.$$

Hence if v be the same—or the combination be achromatic for the two colours to which these refractive indices belong,—we must have

$$0=\frac{d\mu_1}{\mu_1-1}\cdot\frac{1}{f_1}+\frac{d\mu_2}{\mu_2-1}\cdot\frac{1}{f_2},$$

which is the condition of achromatism.

COR. If there be n lenses in contact, the condition of achromatism will be

$$0 = \Sigma\left(\frac{d\mu}{\mu - 1} \cdot \frac{1}{f}\right)$$

As in the case of the prisms Art. (174), we see that this equation can be satisfied for $n - 1$ systems of values of $d\mu$, treating the focal lengths of the lenses as the unknown quantities, so that the combination can be made to unite n colours *or fixed lines* of the spectrum.

The *ratios* of the focal lengths being determined by the $n - 1$ equations of condition of achromatism, the focal lengths can be completely determined if the focal length of the combination for a given kind of light be given.

N.B. We may also remark that so far as achromaticity is concerned the *order* in which they are placed is immaterial.

179. *A pencil of parallel rays is refracted directly through two thin lenses, on the same axis, separated by a given interval; to find the condition of achromatism.*

If f_1, f_2 be the focal lengths of the lenses for a *fixed line* whose index in them is μ_1, μ_2 respectively; a the distance of their centres:—the distance v from the second lens of the principal focus of the combination for this kind of light is given by the equation

$$\frac{1}{v} = \frac{1}{f_1 + a} + \frac{1}{f_2} \text{ (Art. 114, Cor.).}$$

Now if μ_1 become $\mu_1 + d\mu_1$, the corresponding change in

$$\frac{1}{f_1 + a}, \text{ i.e. in } \frac{\frac{1}{f_1}}{1 + \frac{a}{f_1}}, \text{ is equal to}$$

$$\frac{\frac{1}{f_1}\left(1 + \frac{d\mu_1}{\mu_1 - 1}\right)}{1 + \frac{a}{f_1} + \frac{a}{f_1} \cdot \frac{d\mu_1}{\mu_1 - 1}} - \frac{\frac{1}{f_1}}{1 + \frac{a}{f_1}}.$$

If this be expanded in powers of the small quantity $\frac{d\mu_1}{\mu_1 - 1}$, and powers higher than the second be neglected, this becomes

$$\frac{f_1}{(a+f_1)^2} \cdot \frac{d\mu_1}{\mu_1 - 1} - \frac{af_1}{(a+f_1)^3}\left(\frac{d\mu_1}{\mu_1 - 1}\right)^2,$$

and if μ_2 become $\mu_2 + d\mu_2$,

$$\text{the alteration of } \frac{1}{f_2} \text{ is } = \frac{d\mu_2}{\mu_1 - 1} \cdot \frac{1}{f_2}.$$

Hence if v remain the same for the two pairs of refractive indices which we have considered,—or if the combination be achromatic for the two corresponding colours,

$$0 = \frac{f_1}{(a+f_1)^2} \cdot \frac{d\mu_1}{\mu_1 - 1} - \frac{af_1}{(a+f_1)^3}\left(\frac{d\mu_1}{\mu_1 - 1}\right)^2 + \frac{1}{f_2} \cdot \frac{d\mu_2}{\mu_2 - 1},$$

the condition of achromatism.

180. *A pencil passes excentrically through two thin lenses separated by a given interval,—its axis before incidence intersecting the common axis of the lenses in a given point :—to find the condition of achromatism.*

Let f_1, f_2 be the focal lengths of the lenses for a *fixed line* whose refractive index in them is μ_1, μ_2 respectively, a the distance between the centres of the lenses. Let the axis of the pencil of this colour before and after refraction through the first lens cut the axis of the lenses at distances b_1, c_1 from the centre of that lens,—and before and after refraction through the second lens at distances b_2, c_2 from the centre of that lens;—also let ϵ, η be its inclination to the axis of the lenses before refraction and at emergence.

Then using first approximations (Art. 137),

$$\frac{1}{c_1} - \frac{1}{b_1} = \frac{1}{f_1},$$

$$\frac{1}{c_2} - \frac{1}{b_2} = \frac{1}{f_2},$$

$$b_2 = c_1 + a,$$

$$\frac{\tan \eta}{\tan \epsilon} = \frac{b_1 b_2}{c_1 c_2};$$

and $\therefore \frac{b_1}{c_1} = 1 + \frac{b_1}{f_1}, \quad \frac{b_2}{c_2} = 1 + \frac{c_1 + a}{f_2};$

$$\therefore \frac{\tan \eta}{\tan \epsilon} = \frac{b_1 b_2}{c_1 c_2}$$

$$= \frac{b_1}{c_1} + \frac{b_1}{f_2} + \frac{a}{f_2} \cdot \frac{b_1}{c_1}$$

$$= 1 + \frac{b_1 + a}{f_2} + \frac{b_1}{f_1} + \frac{ab_1}{f_1 f_2}.$$

Now if $\left.\begin{matrix}\mu_1\\ \mu_2\end{matrix}\right\}$ become $\left\{\begin{matrix}\mu_1 + d\mu_1\\ \mu_2 + d\mu_2\end{matrix}\right.,$

the alteration of $\left.\begin{matrix}\frac{1}{f_1}\\ \frac{1}{f_2}\end{matrix}\right\}$ is $\left\{\begin{matrix}\frac{d\mu_1}{\mu_1 - 1} \cdot \frac{1}{f_1}\\ \frac{d\mu_2}{\mu_2 - 1} \cdot \frac{1}{f_2}\end{matrix}\right.;$

and of $\frac{1}{f_1 f_2}$ is $= \frac{1}{f_1 f_2}\left\{1 + \frac{d\mu_1}{\mu_1 - 1} + \frac{d\mu_2}{\mu_2 - 1}\right\} - \frac{1}{f_1 f_2}$

$$= \frac{1}{f_1 f_2}\left\{\frac{d\mu_1}{\mu_1 - 1} + \frac{d\mu_2}{\mu_2 - 1}\right\},$$

neglecting the product of $\frac{d\mu_1}{\mu_1 - 1}$ and $\frac{d\mu_2}{\mu_2 - 1}$.

Hence, if η be the same for the two pairs of refractive indices, or the combination be achromatic for the two corresponding kinds of light,

$$0 = \frac{b_1 + a}{f_2} \cdot \frac{d\mu_2}{\mu_2 - 1} + \frac{b_1}{f_1} \cdot \frac{d\mu_1}{\mu_1 - 1} + \frac{ab_1}{f_1 f_2}\left(\frac{d\mu_1}{\mu_1 - 1} + \frac{d\mu_2}{\mu_2 - 1}\right) \ldots (A),$$

the condition of achromatism.

181. COR. If the lenses be of the same substance, or $\mu_1 = \mu_2$, the condition becomes

$$0 = \frac{b_1 + a}{f_2} + \frac{b_1}{f_1} + \frac{2ab_1}{f_1 f_2},$$

$$\text{or } 0 = f_1 + f_2 + 2a + \frac{af_1}{b_1};$$

$$\therefore a = -\frac{f_1 + f_2}{2 + \frac{f_1}{b_1}}.$$

If b_1 be very large, we have approximately,

$$a = -\frac{f_1 + f_2}{2} \quad \text{.......................} (B).$$

In this case condition (A) is satisfied for all corresponding values of $\frac{d\mu_1}{\mu_1 - 1}$ and $\frac{d\mu_2}{\mu_2 - 1}$, or the combination is achromatic for all kinds of light,—as might have been foreseen, since the lenses being of the same substance *irrationality* has here no existence.

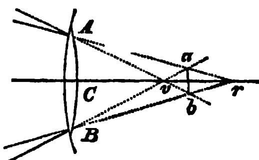
The relation (B) obtains in the combination of two lenses, known as Huyghens' Eye-piece (Art. 140), so that by a happy coincidence the relation between the lenses selected by Huyghens has the great additional advantage of being achromatic—or very nearly so.

For further information respecting achromatic combinations of lenses, the student may consult some memoirs in the *Phil. Trans.* 1821; *Cambridge Phil. Trans.* Vol. II. and III.

182. *Chromatic Aberration.*

A pencil of compound rays is refracted directly through a thin lens; to find the chromatic aberration.

Let C be the centre of a thin lens AB through which a pencil is directly refracted, v, r the geometrical foci for the most and least refracted rays of the pencil, μ_v, μ_r the indices of refraction for the same respectively, μ the index for mean rays, v the distance from the centre of the geometrical focus of mean rays, u the distance from the centre of the origin of the pencil which we suppose to consist of white light at incidence, r, s the radii of the lens,—lines as usual being accounted positive when measured from the centre contrary to the direction of the incident pencil. Then



$$\frac{1}{Cv} - \frac{1}{u} = (\mu_v - 1)\left(\frac{1}{r} - \frac{1}{s}\right),$$

$$\frac{1}{Cr} - \frac{1}{u} = (\mu_r - 1)\left(\frac{1}{r} - \frac{1}{s}\right);$$

$$\therefore \frac{Cr - Cv}{Cr \,.\, Cv} = (\mu_v - \mu_r)\left(\frac{1}{r} - \frac{1}{s}\right).$$

Since $Cr - Cv$ is usually small compared with Cv or Cr, we may put

$$Cr \,.\, Cv = v^2, \text{ nearly,}$$

and $$(\mu_v - \mu_r)\left(\frac{1}{r} - \frac{1}{s}\right) = \frac{\mu_v - \mu_r}{\mu - 1} \,.\, (\mu - 1)\left(\frac{1}{r} - \frac{1}{s}\right) = \frac{\varpi}{f},$$

if ϖ be the dispersive power of the lens (Art. 171) and f its focal length for mean rays;

$$\therefore \text{chromatic aberration} = Cr - Cv = rv = \frac{\varpi v^2}{f}.$$

COR. Of the two points, r, v,—v is the nearer to C, *except* when the origin of the pencil is between the centre of the lens and the principal focus of rays incident in the contrary direction.

183. *To find the magnitude of the circle of chromatic aberration of a pencil refracted directly through a thin lens.*

If spherical aberration be neglected, v and r are the points of divergence or convergence of the most and least refracted rays. If then A, B be the extremities of the pencil in the plane ABv, and if Ar cut Bv produced in a, and Br cut Av produced in b, then,—AC being equal to BC,—the circle whose diameter is ab and plane perpendicular to Cr is the smallest space through which the whole dispersed pencil passes, and is called the *circle of chromatic aberration.*

Let c the bisection of ab be the centre of this circle, and let $CA = y = CB$.

If the pencil be not very large

$$\text{the triangles } \left.\begin{matrix} vab \\ rab \end{matrix}\right\} \text{ are similar to } \left\{\begin{matrix} vAB \\ rAB \end{matrix}\right. ;$$

$$\therefore \frac{cv}{ab} = \frac{Cv}{AB}, \quad \frac{cr}{ab} = \frac{Cr}{AB};$$

$$\therefore \frac{vr}{ab} = \frac{Cr + Cv}{AB} = \frac{v}{y}, \text{ nearly};$$

$$\therefore ab = \frac{y}{v} \cdot vr = \frac{\varpi y v}{f},$$

which determines the diameter of the circle.

COR. If the incident rays be nearly parallel v is nearly equal to f, and the radius of the circle of chromatic aberration will be independent of the focal length of the lens and vary as its *aperture.*

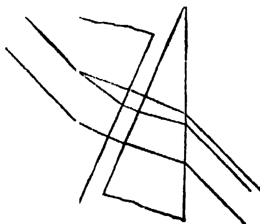
184. In a telescope with a given eye-glass the magnifying power varies as the focal length of the object-glass—(see next chapter),—while the size of the circle of chromatic aberration remains the same if the aperture remains the same. In viewing a white field through the telescope, the circles of chromatic aberration for different points would overlap and form a white field bounded by a coloured border of constant breadth so long as the aperture of the object-glass remains

the same. Hence the greater the magnifying power, the less would be the area of the coloured border in proportion to the whole apparent field of view given by the object-glass. To secure this advantage, astronomers before the discovery of the achromatic object-glass were in the habit of having refracting telescopes of great length,—even as great as 150 feet. Huyghens in particular was celebrated for his skill in making large glasses. (*Herschel, Art.* 458.)

Since the discovery, however, of the means of constructing a compound achromatic object-glass, the necessity for such large instruments has been obviated, and they are now constructed of much more moderate dimensions, and the size of the instrument reduced within a more manageable compass. As we have seen (Art. 178) the conditions of *achromatism* depend only on the *focal lengths* of the component lenses—not at all on their *forms* or the *order* in which they are placed. By a suitable arrangement of these latter quantities—i.e. *forms* and *order*—the conditions requisite for the destruction of spherical aberration (before referred to, Art. 130) can be secured, and a compound object-glass constructed which shall be at the same time both *aplanatic* and *achromatic.*

185. *Colours of natural bodies—Primary colours.*

The colours of the spectrum given by a prism—as in Newton's experiment—could again be combined into white light by an equal prism set in the opposite direction—or any one or more colours might be insulated (as in fig. Art. 157) and separately examined or combined. From experiments of this kind it appears that *all* the colours of the spectrum must be combined in their natural proportions in order to produce *white* light—but that any shade of colour in nature may be imitated by a combination of the tints of the spectrum with a brilliancy unequalled by any artificial colouring.



If any substance be exposed in the prismatic spectrum, it appears to be of the same colour as that part of the spectrum in which it happens to be placed, but its hue is much more brilliant and vivid when placed in a part of the spectrum analogous to its own natural colour. Hence we are led to conclude that the colours of natural bodies are not qualities inherent in themselves, but arise from their aptitude to absorb and reflect different kinds of light. Thus a substance appears *green*—suppose—in consequence of its disposition to reflect green rays alone, the other prismatic colours of which the white solar light falling upon it is composed, being stifled or absorbed by it..

In forming his theory of the composition of colours, Mayer regarded all colours as resulting from the combination of *three primary colours,—red, yellow* and *blue.* And he gave as results of his experiments the different proportions in which these colours entered into combination to produce others. Dr Young assumed *red*, *green* and *violet* as primary colours, and states that *white* light is composed of these *complementary* colours in the proportions of two parts of red, four of green, and one of violet. It has been shewn by Helmholtz and Maxwell that the following mixtures of two complementary colours, when brought together into the eye, produce the effect of whiteness;—*violet* and *greenish yellow; indigo* and *yellow; blue* and *orange; greenish blue* and *red.* Sir D. Brewster regards *red, yellow* and *blue* as primary colours, a certain portion of each existing at every point of the spectrum—the colour being determined by the predominating colour at that point, mixed with white light:—but this theory of Brewster's has been proved to be fallacious, for Helmholtz has shewn that the *green* ray, for example, is not made up of *blue* and *yellow* light superposed, and we cannot separate anything else but *green* out of it. Hence we conclude that each particular ray has its own peculiar colour, and that light of each degree of refrangibility is monochromatic.

See Dr Young's *Lectures on Natural Philosophy; Trans. Roy. Soc. Edin.* Vols. XII. XV.; Chevreul, *Cercles Chromatiques,* Helmholtz, *Handbuch der Physiologischen Optik,* in Karsten's *Allgemeine Encyclop. der Physik.*

See also Sir John Herschel's remarks on Newton's Theory of the Colours of Natural Bodies in ART. *Light*, Encyc. Metrop. Arts. 1134 &c.

185*. We give a short table of refractive indices—or values of μ, from *vacuum* into different substances—for rays of *mean* refrangibility. The *fixed line E* of the spectrum, which is in the *green* space and nearly the *mean* ray may conveniently be taken as the *line* or *colour* of reference.

Substance		Index
Vacuum		1·000000
Atmospheric air *at freezing temperature and pressure* $= 29^{in}\cdot922 = 0^{m}\cdot7$ *according to* BIOT		1·000294
Ice	*from* 1·3070 *to*	1·3100
Water, *fresh:* see p. 154		1·336
,, *salt* (? *sea*)		1·343
Alcohol (S.G. 0·866)		1·371
Nitric Acid (S.G. 1·48)		1·410
Fluor spar	1·433 *to*	1·436
Oil of Turpentine, S.G. 0·885, *ray E*		1·47835
Alum		1·457
Oil of Olives		1·470
Canada balsam		1·532
Brazil pebble, S.G. 2·62		1·532
Glass, crown	1·525 *to*	1·563
,, plate	1·500 *to*	1·540
,, flint	1·576 *to*	1·642
Glass, *tinged red with gold*		1·715
,, lead 1, flint 2		1·724
,, lead 3, flint 4		1·732
Gum Arabic		1·512
Iceland spar	1·4887 *to*	1·6636
Rock crystal	1·547 *to*	1·562
Topaz	1·6102 *to*	1·652
Ruby	1·601 *to*	1·779
Zircon	1·961 *to*	2·015
Garnet		1·815
Diamond, *various specimens*	2·439 *to*	2·775

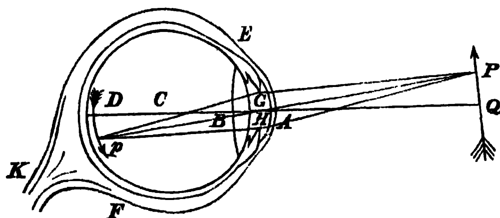
CHAPTER X.

OF VISION AND OPTICAL INSTRUMENTS.

186. We proceed to give a description of the optical structure of the eye, and the manner in which vision takes place—and we shall then go on to describe some of the more important optical instruments.

187. *Description of the Eye.*

The human eye consists of transparent substances enclosed in two coats nearly spherical in form. The figure



represents a section of it by a plane through a line ACD, called the axis of the eye, with respect to which the surfaces of the coats are symmetrical.

The exterior coat EDF, called the *sclerotica*, is horny and opaque,—except the front part A, which is transparent and slightly protuberant beyond the nearly spherical surface of the rest, and called the *cornea*. The second coat, interior to this, is called the *choroides;* it is opaque but has a circular aperture GH behind the cornea,—called the *pupil*—whose centre is in the axis of the eye. Over the back of

the eye there extends a black velvety substance, called the *pigmentum nigrum,* perfectly incapable of reflecting light—and in this is embedded a delicate tissue of nerves called the *retina,* which communicate with the brain by means of the optic nerve K.

B is a soft transparent jelly-like substance—called the *crystalline lens*—in the form of a double convex lens having its axis coincident with ACD the axis of the eye, and held in its place by tendons springing from the choroides. The spaces between the cornea and the crystalline, and between the crystalline and the retina, are filled with transparent fluids, called respectively the *aqueous* and the *vitreous* humours. The refractive index of each of these humours out of air is nearly that of water:—the refractive index of the crystalline is a little greater.

Note. The values given by Sir J. Herschel are for the

aqueous humour	$\mu = 1{\cdot}337$
vitreous humour	$\mu = 1{\cdot}339$
crystalline lens, *mean* . . .	$\mu = 1{\cdot}384$.

The values given by Prof. M^c^Kendrick are for the

aqueous humour, *vitreous* humour . . .	$\mu = \frac{103}{77} = 1{\cdot}3379$
crystalline lens (? *mean*) . .	$\mu = \frac{16}{11} = 1{\cdot}4545$

188. *To explain the manner in which vision takes place.*

Let the axis of the eye be directed to a point of an object PQ. A pencil from any point P falls upon and is refracted by the cornea. Of this pencil a portion limited by the aperture GH is again refracted by the aqueous humour, the crystalline lens and the vitreous humour, and is made to converge very nearly to a point p on the retina. The impression thus made on the retina is communicated to the brain and produces the sensation of vision of the point P.

189. The spot K where the nerves of the retina pass through the coats of the eye, is insensible to vision—and from this cause is called the *blind spot,* or *punctum cæcum.* The following simple experiment is given by Sir J. Herschel for

shewing the existence of this point. "On a sheet of black paper or other dark ground place two white wafers having their centres three inches distant. Vertically above that to the *left* hold the *right* eye at 12 inches from it, and so that when looking down on it the line joining the two eyes shall be parallel to that joining the centres of the wafers. In this situation, closing the left eye and looking full with the right at the wafer perpendicularly below it, this *only* is seen, the other being completely invisible. But if removed ever so little from its place, either right or left, above or below, it becomes immediately visible, and starts as it were into existence. The distances here set down may perhaps vary slightly in different eyes."

189*. The central part of the retina is more sensitive to light than the more anterior parts of it. About the point where the axis of the eye meets the retina is an oblong *yellow spot*, which is the most sensitive to light, and is chiefly employed in distinct vision. This *yellow spot*,—or *punctum luteum*—has a horizontal diameter of about ·08 *inch*, and a vertical diameter of about ·003 *inch*—and corresponds to a visual angle of from 2^0 to 4^0. The central part of the spot, or *fossa*—is believed to possess the most acute sensibility—the diameter of it being only about ·008 *inch*.

The great mobility of the eye enables it to bring the images of successive points of an object on to the more sensitive parts of the retina very quickly;—and as the impression produced by light on the retina continues about $\frac{7}{60}$ *of a second* after the light has ceased to act, by this means a distinct visual picture of an object is acquired.

190. The front part of the choroides surrounding the pupil is called the *uvea* or *iris*, which is differently coloured in different individuals, and by an involuntary action is capable of expanding or contracting the pupil within the limits of about ·25 and ·09 inches:—so as to admit a larger or smaller pencil of light as the object viewed is less or more brilliant. This involuntary action may be observed by any one watching his own eye as he walks towards a mirror, near which is placed a bright object—as a lamp for instance.

The aperture of the pupil in the human eye is always circular—but in animals of the cat kind the vertical diameter appears to remain invariable, the contraction taking place in a horizontal direction,—in the eye of the horse, on the contrary, the horizontal diameter of the pupil remains nearly constant, whilst it admits of contraction in a vertical direction.

The eye is able to adapt itself automatically to objects at different distances, so as to make pencils of different degrees of divergency converge nearly to a point on the retina.

That this focal adjustment of the eye for different distances is effected—in part at least—by a change of form of the crystalline lens,—has been shewn by Prof. Helmholtz;—who by carefully observing the images of a bright object reflected at the anterior and posterior surfaces of this lens, found that as the eye adjusted itself for different distances these images underwent a change in form and position which could only be accounted for by a change of curvature of the surfaces of the lens: and this is probably attended by a change of form in the cornea likewise.

From the accurate measures which have been made it further appears that the surface of the cornea is a prolate spheroid, and the anterior and posterior surfaces of the crystalline lens are oblate spheroids:—the density also of this lens is found to increase from the outside towards the centre, the tendency of which is to correct the aberration by shortening the focal length for rays which pass near the centre. The forms of the several surfaces, as might be expected, vary in the eyes of different animals—in fishes the crystalline lens is nearly spherical.

This power of adaptation however does not enable the eye to see objects within a certain distance—in general about eight inches—but of an object *sufficiently brilliant* at any greater distance distinct vision can be obtained unless the obliquity or divergence be such that the point p does not fall on the retina.

This power of adaptation depends largely on habit. "A North American savage has a most perfect vision of very distant objects, but can hardly distinguish one held within arm's length." *Coddington.*

Rays which are convergent at incidence on the eye cannot be brought to convergence on the retina.

The eye when employed in a natural manner is *achromatic* in a remarkable degree; there is no appearance of coloured fringes about the edges of an object, and in looking at any variegated object the colours do not run into one another, or produce chromatic confusion; and this is true not only for those parts of the object which appear near the centre of the field of view, but also for those which are seen by pencils of considerable obliquity. Experiments however by Wollaston, Fraunhofer, and others have shewn that this achromatism of the eye is not absolutely perfect.

191. *Of Binocular Vision, or vision with two eyes.*

When an object is viewed by both eyes, an inverted image is formed upon the retina of each eye, but only a very small part of the object is seen *distinctly* at one and the same time;—a portion of it, for instance, which would subtend an angle of a few degrees at the centre of the eye. The point at which the axes of the two eyes intersect is the point which is seen most distinctly,—and the extreme mobility of the eyes within certain limits enables them to change this *point of regard* from one part of an object to another with great ease and rapidity. And although we see *distinctly* only a small space at once, yet the visual pictures on the retina give us cognizance of the general relations of surrounding objects to a much larger extent. Thus if we are *regarding* a particular picture in a room we are simultaneously conscious of the relative position of other pictures, the furniture, &c. of the room, and thus a general impression is formed which is rendered more accurate and complete by making the surrounding objects successively *points of regard.*

Further, as the view of external objects presented to either eye is not strictly one and the same, the pictures formed on the retinæ are dissimilar; and this dissimilarity combined with the relative degrees of light and shade in the objects observed, and the variation of inclination of the axes of the eyes, materially assists the mind in forming its impression of *projection* or *solidity*, and of *relative distance*

of the objects contemplated; the mind being prepared by a long course of experience, tactile, muscular and otherwise, to form definite conclusions as to figure, &c. from definite visual sensations.

Thus of the two pictures of a scene prepared for a *stereoscope*, neither *by itself* gives us any impression of solidity; but when they are both viewed through the instrument, the two eyes receive the dissimilar pictures in the same way as if they were actually contemplating the scene portrayed; the illusion is complete, and the figures start into relief as if we were looking at an actual scene in nature.

This result does not follow unless the pictures presented to the two eyes agree (approximately at least) with the pictures which would actually be presented by a real scene; and the difficulty of making the two pictures coalesce is increased when the mind is not familiar with the objects or the kind of objects intended to be represented.

192. For the purpose of shewing the important share which the mind has in interpreting visual sensations, Prof. Wheatstone,—to whom we are indebted for the invention of the stereoscope,—has contrived a *Pseudoscope*, an instrument consisting of a pair of prisms with certain adjustments so that the image thrown upon either retina may be such as, without the intervention of the instrument, would naturally be received by the other. If we put the action of the mind out of consideration, we should expect that everything viewed by this instrument would be turned inside out, *elevation* would become *depression*, and *vice versâ*;—in fact, it ought to give what Prof. Wheatstone calls the *converse of relief*. This effect, indeed, follows easily for objects whose *original* and *converted* forms are familiar to the mind,—as in a seal and its impress,—but in other cases the mind admits the converted form with greater difficulty, according as the converted form is less familiar and probable.

193. It has sometimes been felt as a difficulty, that objects appear *erect*, although the image of them on the retina is *inverted*. *Erect* and *inverted* are simply terms implying *relative* position, and as our optical knowledge of *all* external

objects is derived from the picture of them on the retina, there is the same *relative* displacement of all objects as regards their disposition in space. Thus whilst the picture of men on the retina exhibits them with their heads *downwards*, at the same time heavy bodies appear to fall *upwards*. The mind forms an estimate of all the relative parts of the picture simultaneously, and its relation to external objects is judged of by experience and habit.

As however a discussion of the physiology of vision is beyond our purpose, the student who wishes to pursue the subject may consult an interesting article on *Binocular Vision* in the *Edinburgh Review* for *October*, 1858, and the authorities there referred to. Also *Edinburgh Review* for *October*, 1881, article on "Lectures on the Recent Progress of the Theory of Vision by Prof. Helmholtz," translated by Dr Pye Smith, 1873. Also *Eye-sight, good and bad*, by Robert Brudenell Carter, F.R.C.S., London, 1880.

194. It may be interesting to know the dimensions of the human eye,—they will of course vary in different individuals, but the following are given as belonging to an eye of average size for a person in middle life. The axis of the eye measured from the outer surface of the cornea to the retina = ·95 inches, and the portion of this length occupied by the cornea = ·04 of an inch, the aqueous humour = ·11, crystalline lens = ·17, vitreous humour = ·63.

Interior transverse diameter of the eye	·90	inches.
Vertical chord of the cornea	·46	...
Horizontal chord of the cornea	·49	...
Chord of the crystalline lens	·37	...
Radius of external surface of cornea	·33	...
Radius of anterior surface of crystalline	·33	...
Radius of posterior surface of crystalline	·24	...

The *centre* of the eye for optical purposes is a point nearly in the centre of the pupil, in the plane of the iris.

The angle between the axis of the eye and the line joining the centre of the punctum cæcum with the centre of the eye is about 14°, the breadth of the punctum cæcum is about $\frac{1}{6}$ inch, subtending an angle of 5° at the centre of the eye.

By the revolution of the eye in its socket, its axis has a range of about 55° in every direction about its mean position.

The impression produced by light on the retina continues about $\frac{1}{80}$ *of a second* after the light has ceased to act, so that if a bright object be whirled round in a circle, the period of its gyration being less than this, it will appear as a *continuous* bright circle. With respect to the magnitude of the *minimum visibile*, or apparent magnitude of the least visible object, it is usually stated that an object is invisible to most eyes unless it subtends an angle of at least *one minute.* This however will vary for different eyes, and the *brightness* of the object must materially influence the question, since we find that the fixed stars produce a distinct and vivid impression on the retina, although the angle which they subtend to the eye is so small as to be incapable of being measured by the best instruments.

If a series of equidistant holes be pierced along the circumference of a disc of cardboard, which is made to turn about its centre, and we look through these holes, we shall have successive views of exterior objects, very brief and at short intervals: we shall see them in the situations they occupy at the instant they are perceived. If they are fixed, each impression will be identical with the preceding one; we shall see them without displacement and without interruption, only their brightness will be diminished. If they are in motion, we shall perceive them in successive positions, as if they had jumped from one to another.

Suppose, for instance, there is fixed behind the revolving cardboard a figured sheet divided into as many sectors as there are holes—say *six*—through hole *no.* 1 we view *sector no.* 1,—when hole *no.* 2 arrives, the *sector no.* 2 will take the place of the preceding one—and so in succession. In each of these sectors let there be represented, *for example,* a blacksmith whose figure is the same in each, only his arm which holds a hammer is depressed in *no.* 1, a little raised in *no.* 2, still more raised in *no.* 3,—at the highest elevation in *no.* 6. We pass on rapidly to *no.* 1 again—and the effect is that the eye which receives these impressions in

succession seems to see the hammer raised little by little, and suddenly to drop. In a way similar to this may be explained various amusing toys, the *thaumatrope*, *phenakistiscope*, the *wheel of life*, the *anorthoscope*, &c.

If a group of objects be in motion in a dark room,—for example, a picture on a cardboard whirling about an axis—and the darkness be illuminated by an electric spark, the objects do not seem to be in motion, but stationary in the position they occupied at the instant:—shewing that the duration of the illumination is so brief that no sensible change of position of the group takes place during it.

The numerical results of this Article are taken from Lloyd's *Treatise on Light and Vision.* Perhaps the most complete Treatise on the Physiology of the Eye is Helmholtz's *Optique Physiologique*, 1867. The student may also consult generally the article *Eye* by Professor McKendrick, in the *Encyc. Britt.* 9th ed. 1878. Also Professor J. D. Everett's edition of *Deschanel's Natural Philosophy*, 1882, *part* iv. *Optics*:—which contains numerous excellent diagrams.

195. *Defects of Vision.*

An eye which produces too great refraction of a pencil incident upon it brings pencils from distant points to convergence at points so far before the retina as to produce no distinct impression upon it. This defect is called *short sight.*

On the other hand, for an eye which cannot sufficiently refract a pencil, the least distance of distinct vision is greater than eight inches, pencils from points within this least distance being brought to convergence behind the retina. This is *long sight.*

196. Vision through optical contrivances depends on the fact that if a pencil diverging from a given point fall on the eye, it is immaterial whether that point be an actual source of light, or whether the rays have been made to converge to it and afterwards to diverge. An image therefore is visible in the same manner as a luminous object in the same position would be, with this limitation,—that from any point of a luminous object rays diverge in all directions, but from

any point of an image rays diverge only in directions corresponding to the directions of those rays which form that point in the image.

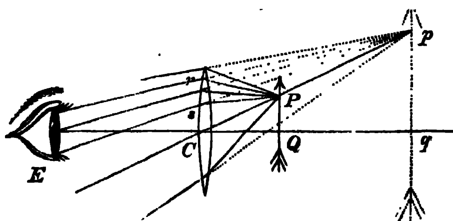
In general explanations of vision through optical instruments, *spherical aberration* may from its smallness be disregarded. Pencils may be considered after reflexion or refraction to converge to or diverge from a point, and an *excentrical* pencil may be supposed to have the same point of divergence or convergence as the centrical pencil from the same origin.

Further, when an object is at such a distance as to be conveniently seen, the pencil received by the eye from any point of it will from its smallness have a *very small* degree of divergence,—so small that for the purpose of general description of an instrument we shall regard the pencils which emerge from the eye-glass to consist of parallel rays.

The slight adjustment requisite to give distinct vision of the image must be performed by each observer for himself.

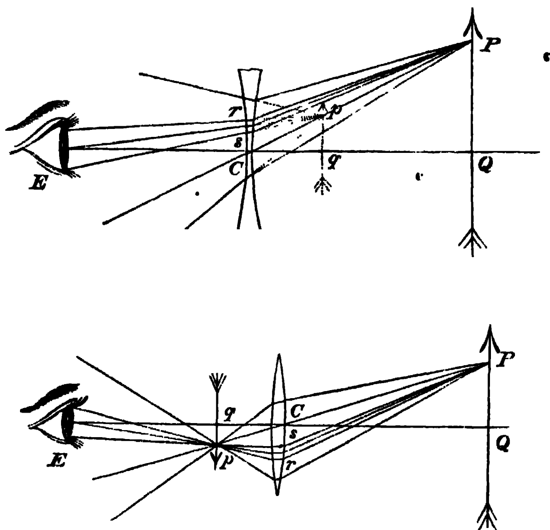
197. *Vision through a lens.*

Let PQ be a small luminous object, C the centre of a lens whose axis is CQ, E the centre of the pupil of an eye whose axis coincides with the axis of the lens. A pencil of



light diverging from a point P of PQ falls upon the lens, and after refraction may be considered as diverging from some point p in CP, or in CP produced, or as converging to some point p in PC produced. Thus pq an image of PQ is formed.

If Eq be not less than the least distance of distinct vision of the eye, then of the pencil diverging from p, the pupil selects

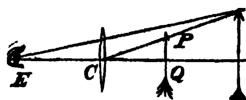


a portion prs which has been refracted *excentrically* through the lens, and by this pencil the point p is visible. Thus the image pq will be seen by the eye.

COR. If PQ be very near to the principal focus of the lens so that the image pq may be very distant, the excentrical pencil pE by which the point p is seen may be considered to consist of rays parallel to pC. (*Compare* Art. 115.)

198. *To find an expression for the visual angle under which a small object is seen through a lens.*

Let EC be the axis of the lens, E the eye, PQ the object, pq its image, $EC = x$, $CQ = u$, $PQ = y$, $EQ = d$, $f =$ focal length of the lens, $\angle\, pEq = \phi$, $\angle\, PEQ = \alpha =$ angle which PQ subtends at the eye.



Then

$$\frac{1}{Cq} = \frac{1}{f} + \frac{1}{u}, \quad \frac{pq}{PQ} = \frac{Cq}{CQ} = \frac{f}{u+f},$$

$$\tan\phi = \frac{pq}{Eq} = \frac{Cq}{Eq} \cdot \frac{PQ}{CQ} = \frac{1}{1 + \frac{x}{Cq}} \cdot \frac{y}{u}$$

$$= \frac{y}{u + x\left(1 + \frac{u}{f}\right)} \quad \text{.............. (i).}$$

Also

$$\tan\alpha = \frac{y}{u+x};$$

$$\therefore \frac{\tan\phi}{\tan\alpha} = \frac{u+x}{u + x\left(1 + \frac{u}{f}\right)} \quad \text{................. (ii).}$$

If pq be at the distance of distinct vision $\Delta = Eq$ suppose, then

$$\frac{1}{\Delta - x} = \frac{1}{f} + \frac{1}{u} \quad \text{....................... (iii).}$$

The results (i), (ii), (iii) enable us to discuss the apparent magnitude for different relative positions of the object.

If the rays emerge in a state of parallelism, then $u = -f$, the lens must be a convex one—i.e. f negative—and we shall have $\tan\phi = -\frac{y}{f}$ from (i). Hence the less f is, the greater will ϕ be,—and this suggests a reason why $\frac{1}{f}$ is called the *power* of a lens (Art. 104).

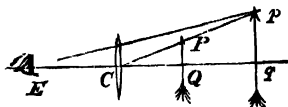
199. In the results of the previous article, if $\phi > \alpha$ the visual angle of the image is greater than that of the object, and the latter may be said to be magnified, and conversely. But it will be useful to compare the magnitudes of the image and object on the supposition that they are both observed at *any the same* distance from the eye—that of distinct vision, for instance—and we give the following *definition*.

DEFINITION. When an object is seen through a lens the magnifying power of the lens is the ratio of the angle which the image subtends at the eye—to the angle which the object would subtend at the eye if it were in the position of the image and viewed directly.

This mode of estimating magnifying power is equivalent to comparing the linear magnitudes of the image and object.

200. *To find the magnifying power of a lens* for given positions of the object with respect to the lens.

The object being supposed small the angle which pq subtends at the eye $= \frac{pq}{Eq}$, and PQ viewed at distance Eq would subtend an angle $= \frac{PQ}{Eq}$. If then m denote the magnifying power of the lens,



$$m = \frac{pq}{PQ} = \frac{Cq}{CQ}. \quad \text{But } \frac{1}{Cq} - \frac{1}{CQ} = \frac{1}{f};$$

$$\therefore m = \frac{v}{u} = 1 - \frac{Cq}{f}.$$

201. *To examine the values of* m *under different circumstances.*

In the figure of last Article

(i) If f be positive—or the lens concave—Cq is positive and the image erect. Also Cq is $< f$, therefore m is < 1, or the image is diminished with respect to the object.

(ii) If f be negative—or the lens convex—Cq is positive or negative, and the image erect or inverted, according as CQ is less or greater than the focal length of the lens without reference to sign.

In the former case m is > 1, or the image is magnified. In the latter case, the image is magnified (i.e. $m > 1$ nume-

rically) if $\frac{Cq}{f} > 2$—or the distance of the object from the lens less than twice the focal length,—otherwise the image is diminished.

202. A convex lens having the effect of producing an erect and more distant image of a near object, assists the eye of a long-sighted person,—and a concave lens by producing an erect and nearer image of a distant object, assists a short-sighted person.

This explains the use of spectacles and eye-glasses. It was remarked by Dr Wollaston that the best form of lenses for this purpose is *concavo-convex*—since the indistinctness for oblique pencils arising from aberration is much less in a lens of such a form than in an equiconvex or equiconcave lens of the same power:—in other words, the field of distinct vision is larger in the former than in the latter.

If an object be viewed through a convex lens,—the object not being farther from the lens than its principal focus—the divergence of the pencil by which any point of it is seen, is greater as the object is nearer to the lens.

Other defects of vision arising from the want of symmetry in the refracting surfaces of the eye, may be corrected or diminished by using eye-lenses of various forms—adapted to each particular case: for example, lenses one surface of which is spherical and the other cylindrical.

Telescopes.

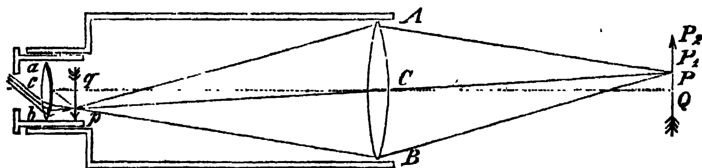
203. When an object is at a great distance the small pencil from any point of it which is selected by the pupil of the eye may not have sufficient illuminating power to make a sensible impression on the retina: and further, the parts of many distant but visible objects cannot be distinguished because the distance between the parts subtends at the eye no appreciable angle. Now if an image of such an object be formed near the observer by a lens or reflector, and pencils converging to, or diverging from, points in this image be refracted through a lens to the eye, the condensation of rays in the pencil from any point of the object may be sufficient to

render that point visible—and the directions of the axes of the pencils from different points may include at the eye appreciable angles. *In other words*, the large lens—or reflector—serves to condense into each point of the image formed by it a larger number of rays than could be received by the unassisted eye, and so makes the image more brilliant than the object;—the eye-glass magnifies this image and also renders a larger extent of it visible at once than could be so without such assistance. Such is the principle of the Telescope.

There are two classes of telescopes—refracting and reflecting telescopes—so named from the manner in which the image of the object viewed is formed;—in the former class the image being formed by a lens, in the latter by a reflector. We will first describe these instruments in their simplest construction—deferring the explanation of the additions and modifications by which the vision of objects through them is improved. *See* Art. 151 *on brightness of images,* also *Art.* 257*.

204. *The Astronomical Telescope.*

ACB is a convex lens,—called the *object-glass,*—fixed in a tube, and *acb* a convex lens,—called the *eye-glass,*—fixed in



another tube which slides in the former—the common axis of the tubes being the common axis of the lenses. The focal length of the object-glass is numerically greater than that of the eye-glass, and when the instrument is in adjustment for viewing very distant objects, the distance between *C, c* the centres of the lenses is the *sum* of their focal lengths.

If the axis of the lenses be directed to a point *Q* of an object *PQ*, which is so distant that a pencil incident on the object-glass from any point of *PQ* may be regarded as con-

sisting of parallel rays—the pencil from a point P after refraction through the object-glass converges very nearly to a point p in PC produced (Art. 197),—Cp being equal to the focal length of the object-glass, and thus pq an inverted image of PQ is formed. This image—from the adjustment of the instrument—is at the principal focus of the eye-glass; and therefore a pencil diverging from a point p of the image consists after excentrical refraction through the eye-glass of rays parallel to pc, (Art. 197), and suitable for giving distinct vision of the image of p formed by the eye-glass, to an eye applied to the eye-glass.

Thus an inverted image of PQ is seen through the Telescope; or rather, the image presents a picture of the object turned *half-round* about the axis of the Telescope (*see* Art. 143).

205. *Field of View*—i.e. *the space that can be viewed with the telescope at one and the same time.*

If P_1 be a point of the object such that the ray P_1A of the pencil from it is refracted in the line Ab, which joins *opposite* parts of the object-glass and eye-glass, then every ray of this pencil—and also every ray of a pencil from any point nearer to Q than P_1 is,—falls upon the eye-glass and is refracted to the eye. Again if P_2 be a point in the object such that the ray P_2B of the pencil from it is refracted in the line Bb, which joins *corresponding* parts of the object-glass and eye-glass, then of this pencil this ray alone falls upon the eye-glass, and is refracted to the eye. Of a pencil from a point between P_1 and P_2 a portion reaches the eye;—which portion is less and less as the point is more and more distant from Q, and no ray of a pencil from a point more distant than P_2 from Q is refracted by the eye-glass.

Hence on looking through the telescope, points whose angular distance from Q exceeds P_1CQ are more and more faint as this distance is greater,—and points whose angular distance from Q exceeds P_2CQ are invisible. This gradual fading away of objects at a distance from the centre of the field is known as the *ragged edge* of the field of view. It is remedied by putting a *stop*—or diaphragm with a circular aperture

—at the position of the image pq, so as to destroy that part of the image which would be seen by partial pencils. The angular extent of the uniformly bright field of view is then the angle subtended at C by the diameter of the aperture of the stop.

If a moving object—as a star moving, say, from *left* to *right*—be observed through the telescope held in a given position—the image of the star traverses the field of view in the opposite direction, from *right* to *left*.

Note. See Art. 257* for some useful remarks on the brightness of images formed by an Astronomical Telescope.

205.* When a small object is placed at a distance from a convex lens equal to the focal length, the cones of rays emanating from the several points of the object are transformed by refraction through the lens into cylinders of parallel rays—and the object viewed through the lens appears to be at a very great distance: such a lens—or combination of lenses—suitably mounted and used for the purpose of producing the same effect as a distant mark is called a *collimator*, and is much employed for adjusting instruments in an observatory.

206. An expression for the radius of the stop (ρ) and the angular radius of the field of view (ϕ) can easily be obtained.

Let $\left.\begin{matrix} f_0 \\ y_0 \end{matrix}\right\} \begin{matrix} f_e \\ y_e \end{matrix}$ be the focal lengths and half-breadths of the object-glass and eye-glass respectively.

Join Ab cutting pq in p—through p draw a line parallel to Cc; then these two lines with the intercepted portions of the lenses—(treated as straight lines),—will give us two similar triangles, whence we have

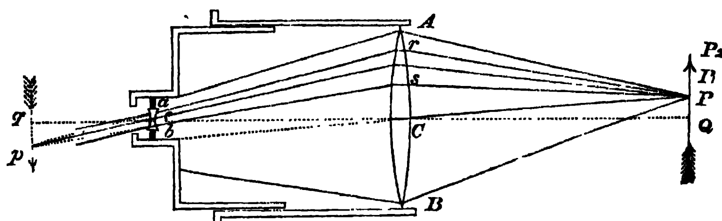
$$\frac{y_e - \rho}{f_e} = \frac{y_0 + \rho}{f_0},$$

$$\text{therefore} \quad \rho = \frac{f_0 y_e - f_e y_0}{f_0 + f_e},$$

$$\text{and} \quad \phi = \frac{\rho}{f_0} = \frac{f_0 y_e - f_e y_0}{f_0(f_0 + f_e)}.$$

207. *Galileo's Telescope.*

ACB is a convex lens called the object-glass, fixed in a tube, and *acb* a concave lens called the eye-glass, fixed in another tube, which slides in the former,—the common axis of the tubes being the common axis of the lenses. The focal length of the object-glass is numerically greater than that of the eye-glass, and when the instrument is adjusted for view-



ing very distant objects, the distance between *C*, *c* the centres of the lenses, is the *difference* of the focal lengths.

If the axis of the Telescope be directed to a point *Q* of an object *PQ*, which is so distant that a pencil incident on the object-glass from any point of it may be considered to consist of parallel rays, the pencil from a point *P* after refraction through the object-glass converges very nearly to a point *p* in *PC* produced,—*Cp* being equal to the focal length of the object-glass—and thus *pq* an inverted image of *PQ* *would* be formed. Of the pencil converging to any point *p* of this image, the eye-glass (which is about the size of the pupil of the eye) selects the portion *prs* which has been refracted *excentrically* at the object-glass,—and since by the adjustment of the instrument *pq* is at the principal focus of the eye-glass, this portion of the pencil after *centrical* refraction through the eye-glass consists of rays parallel to *cp*, and suitable for giving distinct vision of the image of *p* formed by the eye-glass to an eye applied to the eye-glass.

Thus an *erect* image of *PQ* is seen through the Telescope —since the rays from the extremities of the object have not crossed each other before entering the eye.

208. *Field of View of a Galilean Telescope.*

If P_1 be a point in the object such that the ray P_1A of a pencil from it is refracted in the line Aa, which joins *corresponding* parts of the object-glass and eye-glass, then of the pencil from this point, or from any point nearer to Q, a portion falls upon the eye-glass sufficient to fill it. Again if P_2 be a point in the object such that the ray P_2A of the pencil from it is refracted in the line Ab, which joins *opposite* parts of the object-glass and eye-glass, then of this pencil this ray alone falls upon the eye-glass and is refracted to the eye. Of a pencil from a point between P_1 and P_2 the portion which reaches the eye partially fills the eye-glass, and is less and less as the point is more distant from Q; also, no ray of a pencil from a point more distant from Q than P_2 is refracted by the eye-glass. Hence there will be a *ragged edge* to the field of view, which in this Telescope is incurable—because the image formed by the object-glass is *virtual*, and therefore cannot be limited by a stop.

Obs. The vision through the telescope will be most distinct when the refraction through the eye-glass is *centrical*, and hence the size of the eye-glass ought to be nearly the same as that of the pupil of the eye:—if it be much larger and the eye be not applied at its centre, the refraction through it will be excentrical, and the distortion and chromatic dispersion increased.

Note. When an object is viewed through a Galilean telescope, the parts of the image seen will appear in the same relative positions as the corresponding parts of the object:—there is no *inversion* as regards *up* and *down*, nor *reversion* as regards *right* and *left*,—and the *image* of a *moving* object will *move* in the same direction as the object itself *moves*. Hence this telescope is very convenient for observing land objects.

209. *Def.* When an object is viewed by a telescope the *magnifying power* of the telescope is estimated by the ratio of the angle which the image seen subtends at the eye to the angle which the object would subtend at the eye if viewed directly.

210. *To find the magnifying power of the Astronomical Telescope—or of Galileo's Telescope.*

Since the point p of the image pq (figs. Arts. 204, 207) is seen by a pencil whose axis after refraction through the eye-glass is parallel to pc, and the point q by a pencil whose axis is qc, the image of PQ subtends at the eye an angle pcq. And PQ would subtend at the eye an angle PcQ,—or PCQ since the object is very distant.

Hence

$$\text{magnifying power} = \frac{\angle\, pcq}{\angle\, PCQ}$$

$$= \frac{\tan pcq}{\tan pCq} \text{ approximately,}$$

$$= \frac{Cq}{cq}$$

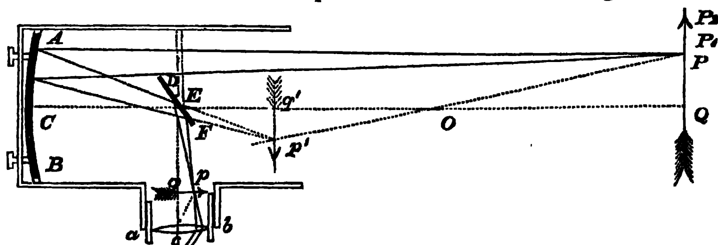
$$= \frac{\text{focal length of the object-glass}}{\text{focal length of the eye-glass}}.$$

Obs. This result gives *approximately* the *linear* magnifying power at any point of the field of view:—though it is strictly true near the centre of the field of view only.

Further, the instruments are supposed to be in adjustment, so that any pencil from a very distant point emerges from the eye-glass as a pencil of parallel rays: *practically* the eye-glass must be pushed in a little, in the case of either telescope, to suit a short-sighted eye,—so that from this cause the magnifying power will be slightly different for different observers.

211. *Newton's Telescope.*

ACB is a concave spherical reflector—or *speculum*—



whose centre is O, and whose axis CO coincides with the axis of the tube at the extremity of which it is placed, DEF a small plane mirror inclined at 45° to the axis of the tube; acb a convex eye-glass placed in a tube which slides in an aperture of the former tube: the axis of the two tubes are at right angles,—the plane of DEF is perpendicular to the plane of the axes of the tubes—and the axis of the lens acb coincides with that of the tube in which it is placed.

If the axis CO be directed to a point Q of an object PQ which is so distant that a pencil incident on ACB from any point of it may be considered to consist of parallel rays, the pencil from a point P after reflexion at this mirror converges very nearly to a point p' in PO produced, Op' being $= \frac{1}{2} CO$ (Art. 115), and there $p'q'$ an inverted image of PQ *would* be formed—or rather, the image $p'q'$ would be a picture of the object turned *half-round* about the axis of the spherical reflector. This pencil being reflected again by DF will converge very nearly to a point p, the length of path of any ray to p being equal to that to p' (Art. 60). Thus pq an inverted image of PQ is formed, and the position of the eye-piece is such that this image may be at its principal focus. Hence the pencil diverging from any point p of the image consists after excentrical refraction through the eye-glass of rays parallel to pc, and suitable for giving distinct vision of the image of p formed by the eye-glass to an eye applied to the eye-glass.

Thus an inverted image of PQ is seen through the telescope.

212. *Field of View* of Newton's Telescope.

If P_1 be a point in the object such that the ray of a pencil from it which, after reflexion at ACB, is incident on the small mirror at D, is reflected by DF in Db the line joining opposite parts of the small mirror and eye-glass, then of a pencil from P_1—or from any point of the object nearer to Q than P_1—every ray which falls on the small mirror is refracted by the eye-glass to the eye. Again if P_2 be a point in the object such that the ray of a pencil from it which is incident on the small mirror at F is reflected in

Fb the line joining corresponding parts of the mirror and eye-glass, then of the pencil from P_2 this ray alone reaches the eye. Points between P_1 and P_2 appear more and more faint the farther they are from Q,—and points at a greater distance from Q than P_2 is are invisible.

213. *Magnifying power* of Newton's Telescope.

Let $p'q'$ (fig. Art. 211) be the virtual image of PQ formed by the large mirror: pq is similar and equal to $p'q'$. Now pq subtends through the eye-glass the angle $\frac{pq}{cq}$, and PQ subtends to the naked eye the angle POQ or $\frac{p'q'}{Oq'}$.

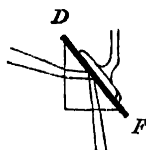
Hence

$$\text{magnifying power} = \frac{pq}{cq} \cdot \frac{Oq'}{p'q'}$$

$$= \frac{Oq'}{cq}$$

$$= \frac{\text{focal length of large reflector}}{\text{focal length of eye-glass}}.$$

Note. The form of the small mirror must be an ellipse in order that it may intercept as little of the incident light as possible, and just reflect all the light incident upon the curved mirror. A rectangular prism of glass is sometimes used instead of a plane mirror; the passage of a pencil through it is indicated in the figure.



213*. To illustrate the use of a Newton's Telescope by an observer—*fig. Art.* 211.—Suppose the axis of the telescope CO to be directed to an object in the *South*—e.g. *a group of stars*—the eye-tube being on the *West* side of the large tube, so that the observer is looking *Eastward* and his *left* hand is towards the *speculum* end of the large tube of the telescope—and his *right* hand towards the *open* end of it. The *virtual* image at $p'q'$ *would* be a picture of the object turned *half-*

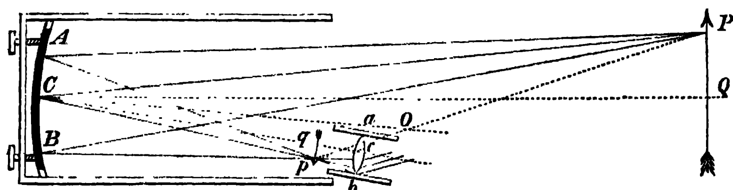
round about the axis of the large tube,—which axis is also the axis of the speculum. This is transferred by the small plane mirror to *pq*—forming there a *real* image: and when viewed by the observer the *East* and *West* parts of the object would be seen *reversed,*—i.e. the *East* and *West* parts of the object would appear towards his *right* and *left* hands respectively—and the picture compared with the object would be *inverted* as to *up* and *down.* If a point of the object move across the field of view—say from *East* to *West,* the corresponding point in the picture will move from *right* to *left* of the observer—i.e. from the *open* end towards the *speculum* end of the large tube, and parallel to its axis.

If the Eye-tube be placed on the *East* side of the large tube—the small plane mirror being suitably adjusted—the observer will now look *West*-ward—that is, he will be turned *half-round* with respect to his former position: a star moving from *East* to *West* will still appear to the observer to move from his *right* to his *left*—i.e. from the *speculum* end towards the *open* end of the large tube.

See a Letter of George Hunt in "*the Observatory,*" No. 57, Jan. 2, 1882.

214. *Herschel's Telescope.*

ACB is a concave spherical reflector, or *speculum,* whose centre is *O,* and whose axis *CO* is inclined at a small angle to the axis of the tube at the extremity of which it is placed, *acb* a convex eye-glass in a sliding tube attached to the inner



surface of the larger tube, the axes of the eye-glass *Cc* and that of the large tube being in the same plane with and equally inclined to *CO.*

If the axis of the large tube be directed to a point *Q* of

an object PQ, which is so distant that a pencil incident on ACB from any point of it may be considered to consist of parallel rays, the pencil from a point P after reflexion at the mirror ACB converges very nearly to a point p in PO produced, Cp being $= \frac{1}{2} CO$ (Art. 115), and thus pq an inverted image of PQ is formed. The position of the eye-glass is such that this image is at its principal focus, and therefore the pencil diverging from any point p of the image consists, after excentrical refraction through the eye-glass, of rays parallel to pc and suitable for giving distinct vision of the image formed by the eye-glass.

This arrangement of the *Eye-glass* and *speculum* makes the Telescope a *front-view* Telescope, and an inverted image of PQ is seen through the telescope—but not reversed as to *right* and *left* with respect to the observer—who is turned *half-round*, so to speak, with respect to the object PQ.

Note. This construction of the reflecting telescope was originally proposed by Le Maire in the early part of last century,—but Sir W. Herschel was the first who made any extensive use of it.

215. *Magnifying power* of Herschel's Telescope.

Since $\angle pCO = \angle PCO$, and $\angle qCO = \angle QCO$ (Art. 8);

$$\therefore \angle pCq = \angle PCQ.$$

Now pq viewed by the eye-glass subtends the angle pcq, and PQ viewed by the naked eye subtends the angle PCQ, or pCq;

$$\therefore \text{magnifying power} = \frac{\angle pcq}{\angle pCq} = \frac{Cq}{cq}$$

$$= \frac{\text{focal length of mirror}}{\text{focal length of eye-glass}}.$$

216. The principle of Herschel's Telescope is the same as that of Newton's—the only object of the plane reflector in the latter being to throw the image unaltered in form into another position where it may be more conveniently viewed. The speculum in Herschel's Telescope is designedly of large

aperture, so that when a faint star is observed a pencil sufficiently large to make it visible may be received by the speculum and reflected to the eye.

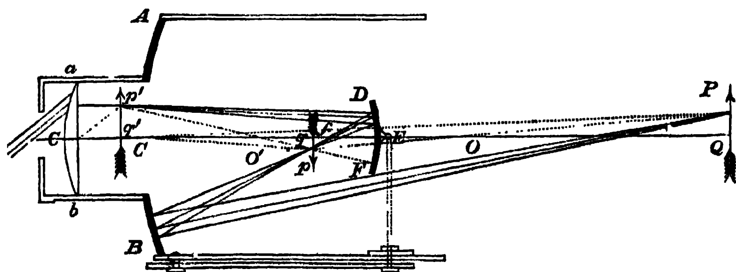
The advantage of Herschel's construction over Newton's arises from there being no part of the incident pencils stopped by the back of the small mirror, and no loss of light from a second reflexion.

The ragged edge of the field of view in Herschel's Telescope may be remedied by a stop placed at the principal focus of the eye-glass—and the angular diameter of the field of view will be the angle which the aperture of the stop subtends at the centre of the face of the speculum—or *approximately*, the angle which the breadth of the eye-glass subtends at the same point.

A similar method of suppressing the ragged edge and estimating the field of view, will apply to Newton's Telescope.

217. *Gregory's Telescope.*

ACB, *DEF* are two concave spherical reflectors, or specula, with a common axis *CE*, which is the axis of a tube at the extremity of which *ACB* is placed,—this mirror being much larger than *DEF* and of larger radius. The concavities of the mirrors are turned towards one another, and the principal focus of *ACB* is between the centre *O'* and principal focus *f* of *DEF*. In a tube which is fixed in an aperture at the centre of *ACB* is a convex eye-glass *acb*, the axis of the eye-glass coinciding with that of the reflectors.



If the axis of the reflectors be directed to a point *Q* of a

very distant object PQ, a pencil from any point of it P after reflexion at ACB converges very nearly to a point p in the straight line produced which joins P with O the centre of ACB, and thus pq a *real* inverted image of PQ is formed at the principal focus of ACB. Since this image is between the centre and principal focus of DEF, a pencil diverging from any point p after excentrical reflexion at DEF converges to a point p' in the straight line produced which joins p with O', the centre of DEF (Art. 115); and thus $p'q'$ an erect image of PQ is formed. This image being at the principal focus of the eye-glass is in a suitable position for being distinctly seen through the eye-glass,—and thus an *erect* image is seen through the telescope.

The parts of the image seen will appear in the same relative positions as the corresponding parts of the object:—there is no *inversion* as regards *up* and *down*, nor *reversion*, as regards *right* and *left*:—and the *image* of a *moving* object will *move* in the same direction as the object itself *moves*. Hence this telescope is a convenient one for observing land objects. Compare *Note*, p. 198.

Note. The *relative* position of the principal foci of ACB and DEF is determined by a consideration of the relation

$$\frac{1}{v}+\frac{1}{u}=\frac{2}{r},$$

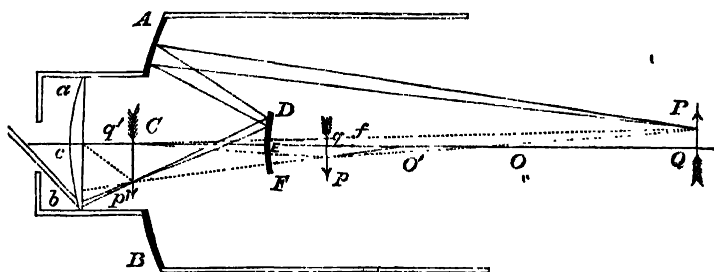
for since it is requisite that Eq' (i.e. v) should be large, $\frac{2}{r}-\frac{1}{u}$ must be small and positive, i.e. Eq must be a little greater than Ef.

Obs. In Gregory's and Cassegrain's telescopes the adjustment for different eyes is performed by shifting the small mirror by means of a screw, as is shewn in the figure,—in each of the other telescopes, it is performed by moving the eye-glass backward or forward.

218. *Cassegrain's Telescope.*

ACB is a concave and DEF a convex spherical reflector with a common axis CE, which is the axis of a tube at one

extremity of which *ACB* is placed. The mirror *ACB*, which is much larger and of greater radius than *DEF*, has its con-



cavity turned towards the convexity of *DEF*, and the principal focus of *ACB* is between *E* and *f* the principal focus of *DEF*. In a tube which is fixed in an aperture at the centre of *ACB* is a convex eye-glass *acb*, the axis of the eye-glass being that of the reflectors.

If the axis of the reflectors be directed to a point *Q* of a very distant object *PQ*, a pencil from any point *P*, after reflexion at *ACB*, converges very nearly to a point *p* in the line which joins *P* with *O* the centre of *ACB*, and then *pq* an inverted image of *PQ* would be formed at the principal focus of *ACB*. Since this image is between *DEF* and its principal focus, the pencil converging to any point *p* after excentrical reflexion at *DEF* converges to a point *p′* in the straight line produced which joins *O′* the centre of *DEF* with *p*, and *p′q′* an inverted image of *PQ* is formed. This image being at the principal focus of the eye-glass is in a suitable position for being distinctly seen through the eye-glass, and thus an *inverted* image is seen through the telescope.

Note. The relative position of the principal foci of *ACB* and *DEF* is determined by a consideration similar to that of Art. 217. Note.

The remarks on the image seen through an Astronomical Telescope,—(p. 195, line 13...p. 196, line 6,)—apply to the image seen through a Cassegrain's Telescope.

219. *Magnifying power* of Gregory's Telescope.

Take the figure of Art. 217.

Let $\left.\begin{array}{l} Cq = Oq = F \\ Ef = O'f = f \end{array}\right\}$ = focal lengths of $\left\{\begin{array}{l}\text{large}\\ \text{small}\end{array}\right.$ mirror.

$cq' = f_e$ = focal length of the eye-glass, numerically.

$qf = x$ = distance between the principal foci of the two mirrors.

Let Q be the point of the object to which the axis of the Telescope is directed,—the image of any other point P by reflexion at ACB is at p—and the image of p formed by reflexion at DEF is at p'—the straight line pp' passing through O' the centre of DEF.

The portion PQ of the object would subtend at the eye the angle POQ, and the image of PQ subtends at the eye the angle $p'cq'$. Hence

$$m = \text{magnifying power} = \frac{\angle\, p'cq'}{\angle\, POQ} = \frac{\tan p'cq'}{\tan POQ} \text{ approximately,}$$

$$= \frac{p'q'}{cq'} \cdot \frac{Oq}{pq} = \frac{Oq}{cq'} \cdot \frac{p'q'}{pq}.$$

$$\text{But } Oq = Cq = F,\ cq' = f_e,\ \frac{p'q'}{pq} = \frac{O'q'}{Oq},\quad O'q = f - x,$$

$$\text{and } \frac{1}{Eq'} + \frac{1}{Eq} = \frac{1}{f},$$

$$\text{or } \frac{1}{O'q' + 2f} + \frac{1}{f + x} = \frac{1}{f}, \text{ whence } O'q' = \frac{f(f - x)}{x};$$

$$\therefore \frac{p'q'}{pq} = \frac{f}{x},$$

$$\text{and } m = \frac{F \cdot f}{f_e \cdot x}.$$

Obs. This expression is strictly accurate at the *centre of the field only,* but may be taken as approximately true over the whole.

Note. The expressions for m in this article and the following one are equally true for Cassegrain's Telescope.

As the adjustment of the Telescope for different eyes is effected by moving the small mirror by means of a tangent screw, the above expression is a very convenient one, since it involves x,—the variable quantity in the instrument,—in a very simple form.

220. An approximate expression for the *linear* magnifying power may be obtained independent of x.

Since Cq' is in general small compared with Oq, we will regard q' as coincident with C, approximately.

We then have $\frac{1}{EC} + \frac{1}{Eq} = \frac{1}{f}$,

$$\text{or } \frac{1}{F+f+x} + \frac{1}{f+x} = \frac{1}{f},$$

$$\text{whence } F+f+x = \frac{f(f+x)}{x} = \frac{f^2}{x} + f;$$

$$\therefore Fx + x^2 = f^2.$$

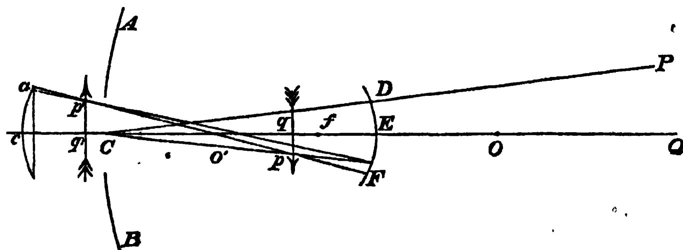
If x^2 be neglected as being small compared with the other terms of this equation, we have $x = \frac{f^2}{F}$;

$$\therefore m = \frac{F \cdot f}{f_e \cdot x} = \frac{F^2}{f_e \cdot f},$$

a result which in general is sufficiently approximate.

221. *Field of view* of Gregory's Telescope.

The field of view may be limited either by the eye-glass or the small mirror.



The following approximate expressions will be practically sufficient.

(i) Suppose the field limited by the eye-glass.

Through O' the centre of DEF and a the edge of the glass draw the line $aO'p$ cutting the image of PQ formed by the large mirror in p—and let PC be the ray which *would* be reflected at C the centre of ACB to p. Then of the pencil from P which fills the large mirror more than half falls on the eye-glass and is refracted to the eye.

Let $\phi = pCE = PCQ =$ angular radius of field of view thus defined,

$ac = y_e =$ half breadth of eye-glass, f_e its focal length,

$EF = y_m =$ .... small mirror,

$Cq = F$, $Ef = f$, $fq = x = \frac{f^2}{F}$, approximately (Art. 220).

$$\text{Now} \quad \frac{\phi}{\angle aO'c} = \frac{\sin\phi}{\sin CO'p} = \frac{O'p}{Cp}$$

$$= \frac{O'q}{Cq} = \frac{f-x}{F}, \text{ nearly.}$$

$$\text{Also } \angle aO'c = \frac{y_e}{O'c} = \frac{y_e}{f_e + O'q'},$$

$$\text{but } O'q' = \frac{f(f-x)}{x} = \frac{F}{f}(f-x) \qquad \text{(Art. 219);}$$

$$\therefore \phi = \frac{f-x}{F} \cdot \frac{y_e}{f_e + \frac{F}{f}(f-x)} = \frac{f \cdot y_e}{F(F+f_e)}, \text{ nearly,}$$

since x is small compared with F and f.

(ii) If the field be limited by the small mirror—let the ray PC after reflexion at C fall upon the edge of the small mirror, then of the pencil which falls upon the object-mirror from any point nearer to Q than P is, more than one-half reaches the eye.

If $\phi_1 = PCQ$ be the angular radius of the field of view thus defined, we have

$$\phi_1 = \frac{y_m}{F+f+x} = \frac{y_m}{F+f}, \text{ nearly.}$$

COR. If the breadths of the eye-glass and small mirror are such that the field of view limited by them severally is the same, then $\phi = \phi_1$ and

$$\frac{y_m}{F+f} = \frac{f \cdot y_e}{F(F+f_e)};$$

$$\therefore \frac{y_m}{y_e} = \frac{f(F+f)}{F(F+f_e)} = \frac{f}{F}, \text{ nearly,}$$

since f_e and f are small compared with F.

222. The telescopes have here been described in their simplest forms, for the purpose of explaining the principle of their construction. We proceed to notice briefly the defects of such instruments which render their modification by compound object-glasses and eye-glasses necessary, whereby these defects are diminished while the principle of the telescope is unchanged.

223. I. *The Astronomical Telescope.*

(i) Let light be considered homogeneous. A pencil from any point after oblique centrical refraction through a single object-glass converges to two focal lines, and the image—or assemblage of circles of confusion—is *indistinct* and *curved* with its concavity towards the object-glass. A direct pencil also is refracted with *aberration.* The compound object-glass commonly used consists of lenses in contact,—and therefore by no arrangement of their *forms* can the indistinctness and curvature of the image be diminished (Art. 146). All that can be done therefore is to construct the lenses so as to produce the least possible aberration in a direct pencil of parallel rays.

If however a distinct and flat image were formed by the object-glass, yet this image viewed through a single lens by

excentrical pencils, would be *indistinct*, *curved*, and also *distorted* (Art. 147). These defects are lessened by properly adjusting the forms of two or more lenses which form a compound *eye-piece*, or *ocular*. The three defects cannot be entirely removed together;—each therefore is diminished as far as possible according to the artist's judgment, and with reference to the use for which the telescope is intended.

(ii) Let light be considered as composed of different species, and let spherical aberration be disregarded (Art. 172). A pencil of such light refracted centrically through a simple object-glass is divided into pencils converging to a series of points in their common axis, and thus a series of coloured images differing slightly in position is formed. The most vivid of these images are united by a compound object-glass of two or more lenses, the focal lengths of which are properly taken.

Again, an achromatic image viewed by a single eye-lens will from unequal refrangibility be confused,—and the confusion will be of a worse kind than that produced by the object-glass, because in the latter case the coloured pencils from the same point have a common axis,—but in the present case the refraction being excentrical, they have not a common axis, and the coloured points corresponding to any point of the image formed by the object-glass are spread over the field. To remedy this confusion the focal lengths of the lenses forming a compound eye-piece are so adjusted that the axes of pencils of the most vivid colours belonging to the same point of the object emerge to the eye parallel to one another,—in which case such pencils, if they be small, affect the eye in the same way as if they were coincident.

Obs. It is worthy of notice that the conditions of *achromatism* affect the *focal lengths* of the lenses combined in an object-glass or eye-piece:—the conditions of diminished *indistinctness*, *curvature*, and *distortion* of the image have reference to the *forms* of the lenses.

See *Astron. Notices*, Vol. XXVIII. p. 202; Vol. XXIV. p. 195; Vol. XXV. p. 22.

See an article in the *Times* Newspaper of March 21, 1881,

for a popular account of the large Refracting Telescope lately made by Mr Grubb of Dublin for the Observatory of Vienna.

224. II. *Galileo's Telescope.*

In the Astronomical Telescope the refraction through the object-glass is centrical,—through the eye-glass excentrical; but in Galileo's the reverse is the case. Hence what has been said of the defects of a simple object-glass and eye-glass in an Astronomical apply respectively to the eye-glass and object-glass of a Galilean telescope. In this telescope, further, the chromatic dispersion through the object-glass is more unpleasant than that of the eye-glass,—and the eye-glass produces distortion in the image.

225. III. *The Reflecting Telescope.*

In the Reflecting Telescope there is spherical aberration from the curved reflectors which produces indistinctness and curvature of the image,—and in the telescopes of Gregory and Cassegrain distortion,—in consequence of the excentrical reflexion at the second mirror. These defects are found to be lessened if the large reflectors be made not exactly spherical, but figures generated by a conic section about its axis—parabolic or slightly hyperbolic. The defects of the single eye-lens which are lessened by a compound eye-piece are the same as have been mentioned in the *Astronomical Telescope.*

226. In the Astronomical Telescope since the pencils pass centrically through the object-glass and excentrically through the eye-glass, the *field of view* depends only on the aperture of the eye-glass;—the aperture of the object-glass affecting only the *brightness* of the field.

In Galileo's Telescope, on the contrary, where the refraction through the object-glass is excentrical, the *field of view* depends upon the aperture of the object-glass—and this is the reason why this telescope is not so generally used for astronomical purposes as for a perspective or opera-glass where small magnifying power is required. For with a high magnifying power, and a field of any considerable extent, the extreme pencils would be refracted by the object-glass at such

a distance from its axis as to make their chromatic dispersion unpleasant and with difficulty diminished. Moreover, the brightness of the field in Galileo's telescope is nearly the same as that of the object, for (fig. Art. 207),

the breadth of the visual pencil at the object-glass rs

: its breadth at the eye-glass

:: focal length of object-glass

: focal length of eye-glass,

i.e. in the ratio of the magnifying power to unity; in other words, the quantity of light received by the eye from any small area of the object is greater than what would be received from the same area without the intervention of the instrument, in the same ratio that the area is magnified—and therefore its apparent brightness is unaffected.

This telescope exhibits objects erect,—which is of great advantage for the purposes for which it is generally employed.

227. *Specula for Reflecting Telescopes.*

The alloy used as a speculum metal is composed of 126·4 parts of copper to 58·9 of tin—from which proportions however different experimenters deviate more or less.

The compound is very brittle and the casting, grinding and polishing specula of large size are matters requiring great care and judgment. The action of the atmosphere upon them for two or three years reduces their brilliancy so much as to render it necessary to re-polish them at such short intervals. In the celebrated Telescope of Lord Rosse, the specula are made of the alloy above mentioned and are of 6 feet aperture and 53 feet focal length—being mounted so as to admit of being used, by a slight adjustment, either on the Newtonian or Herschelian principle.

Dr Steinheil states that he has recently discovered a method of coating a glass mirror with *silver* and polishing the metal side, so as to obtain a metallic speculum of greater reflective power than any hitherto known: according to his estimate the following is the comparative brightness for light

reflected at an angle of 45°—direct light being considered = 100.

	Brilliancy.	Loss of Light per cent.
Direct light	100	0·0
Silver mirror	91	9·
Metallic mirror, Lord Rosse's alloy	67·18	32·82
Object-glass by Fraunhofer, transmitted light	76	24·

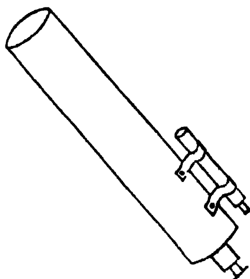
See *Astron. Notices*, Vol. XIX. p. 56, Vol. XXVI. p. 77. The student may also consult Lord Rosse's memoirs in the *Phil. Trans.* for 1840 and 1850. Also a *Description* by Mr Lassell *of a machine for polishing specula, &c. with an account of a 20 feet Newtonian Telescope*, &c. in the *Memoirs of the Roy. Astr. Soc.* Vol. XVIII. See also the Article *Telescope* in Rees' *Cyclopædia* and in the *Encyc. Britann.*

The reflecting telescope constructed by Newton was of very small dimensions,—the speculum being only of about 1 inch aperture, and the focal length 6 inches, with a magnifying power of 38. It is still preserved in the apartments of the Royal Society.

228. In the Gregorian Telescope it is essential that the small mirror should work truly on the axis of the large one, an adjustment which it costs some trouble to secure, and which is very easily deranged; it is also readily affected by a very slight tremor of the stand of the telescope:—but notwithstanding these disadvantages it is in great favour with many observers from its compactness and the convenience resulting from its shewing objects erect.

The Cassegrain construction is not much used:—it is however the construction employed in the large reflector of four feet aperture recently erected at Melbourne Observatory, Australia—made by Mr Grubb of Dublin.

229. A large telescope has generally a small field of view, and there is often difficulty in directing it to a proposed object: to diminish this, a small telescope, called a *finder*, of small power and large field of view is often attached externally to the tube of the large telescope near the eye-end of it; the axes of the two being parallel. A considerable extent of space being thus within the field of the finder the instrument can be moved till the object proposed coincides with the centre of the field of the finder,—i.e. at the intersection of its cross-wires—and it is then at the centre of the field of the large one.



230. *Object-glasses.*

The compound object-glass commonly used in a refracting telescope consists of a lens of crown glass in contact with a lens of flint glass. The conditions which the combination has to fulfil are that it shall be *achromatic* for given kinds of light —(see Art. 178)—and also *aplanatic* or free from spherical aberration. (See Art. 132.) (*Astron. Notices*, Vol. XXIV.)

Achromatic object-glasses have also been constructed of three lenses in contact, consisting of a concave lens of flint glass between two convex lenses of crown glass. The conditions of achromatism can thus be satisfied for more species of light than in the former case: the forms of the lenses will be determined on principles similar to those just referred to. Such object-glasses are now not frequently employed in consequence of the difficulty of centering the lenses so that their axes may exactly coincide. Object-glasses of three lenses have occasionally been made in which the middle one consisted of some liquid, but they have been found not to be very durable in consequence of a chemical change in the liquid.

Besides the references given in this and the preceding article the student may consult *Lehrbuch der Analytischen Optik* von I. C. E. Schmidt, for a method proposed by Gauss

for taking into account the *thickness* of the lenses in a large object-glass,—and Grunert, *Optische Untersuchungen*, also *Camb. Phil. Trans.* Vol. VI.

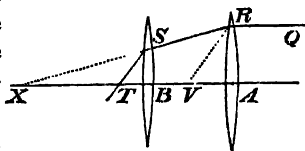
231. *Eye-pieces, or Oculars.*

The defects of a single eye-lens—referred to in Art. 223, viz. indistinctness, linear distortion, as well as that of curvature, and chromatic dispersion—are diminished by employing a compound eye-piece. Those in most general use are (i) *Huyghens'* Eye-piece, (ii) *Ramsden's* Eye-piece: they are also known as the *negative* and the *positive* eye-pieces respectively.

From the expression for tan η (Art. 137) we see that the distortion produced by a single lens $\propto \frac{1}{f^3}$. With a view of diminishing the distortion, Huyghens proposed to construct an eye-piece of two separated convex lenses—dividing equally between the two the deviation produced in an excentrical pencil.

232. *To find the distance between two lenses in order that an excentrical pencil incident parallel to the axis may suffer an equal amount of deviation at each lens.*

Let $QRST$ represent the course of the *axis* of the pencil, f_1, f_2 the *numerical* focal lengths of the lenses A, B, which we will suppose *convex* ones, $AB = a$.



Then using first approximations only, we have

$$\text{deviation at } R = \angle RXA,$$

$$\text{.............. } S = \angle XST = \angle STB - \angle RXA,$$

if these be equal, we get $\angle STB = 2 \angle SXB$,

$$\text{and the angles being small, } \tan STB = 2 \tan SXB;$$

$$\therefore BX = 2 \,.\, BT.$$

$$\text{Also, } \frac{1}{BT} - \frac{1}{BX} = \frac{1}{f_2}; \quad \therefore BX = f_2.$$

But $$BX = AX - AB = f_1 - a;$$

$$\therefore a = f_1 - f_2 \text{ the distance required.}$$

The construction adopted by Huyghens in consistence with this condition was an eye-piece of two convex lenses whose focal lengths are in the ratio of 3 : 1, the less powerful lens being placed as a *field*-glass—i.e. nearest the object-glass of the telescope—and the distance between the lenses being the difference of their focal lengths.

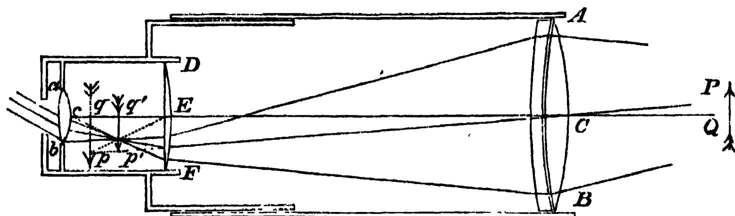
It is a remarkable coincidence—undesigned by the inventor—that if the lenses be of the same material, this construction fulfils simultaneously the condition of achromatism of an excentrical pencil, viz.

$$a = \tfrac{1}{2}(f_1 + f_2) \text{ (Art. 181).}$$

The Huyghenian or *negative*—eye-piece is therefore *achromatic.*

233. *Vision through an Astronomical Telescope with a Huyghens' Eye-piece.*

Let ACB be the object-glass of an Astronomical Telescope directed to a very distant object PQ: DEF the first



lens, or *field-glass*, and acb the second lens, or *eye-glass*, of a Huyghens' eye-piece, the focal length of DEF being three times that of acb, and the distance Ec being the difference, or semi-sum, of the focal lengths without reference to sign. A pencil from a point P of the object after refraction through the object-glass would converge very nearly to a point p, Cp being equal to the focal length of the object-glass,—but being excentrically refracted by the field-glass converges to a point

p' in Ep, and thus $p'q'$ an inverted image of PQ is formed. The position of the eye-piece is such that q' is the bisection of Ec, and therefore this image is at the principal focus of the eye-glass. Hence a pencil from any point p' of the image after excentrical refraction at the eye-glass consists of rays parallel to $p'c$, and suitable for giving distinct vision of the image formed by the eye-glass to an eye applied to the eye-glass. Thus an inverted image of PQ is seen through the telescope.

234. The compensation between the two lenses which renders Huyghens' eye-piece achromatic admits of a simple general explanation.

The deviation of the axis of a pencil of light produced by a convex lens is greater the greater the distance from the axis of the lens at which the axis of the pencil is refracted: for this axis is refracted in the same degree as it would be by a prism whose surfaces touch the lens at the points where the axis of the pencil is incident and emergent, and therefore the deviation is greater as the refracting angle of such prism is greater (Art. 92). Now when a pencil of light refracted by the object-glass falls on the field-glass, it is separated by it into a series of coloured pencils whose axes follow different courses,—the deviation of the axis of the red pencil being least, and that of the violet greatest. The axes of the pencils do not cut the axis of the lenses *between* the lenses, and thus the axis of the red pencil falls on the eye-glass at the greatest distance from the axis of the eye-glass, and consequently is most refracted by it; the axis of the violet falling nearest to the axis of the eye-glass is least refracted by it. Thus the pencils from the same point in the object which are least and most refracted by the field-glass are respectively most and least refracted by the eye-glass, and consequently, by a proper selection of the lenses, may be parallel when they enter the eye.

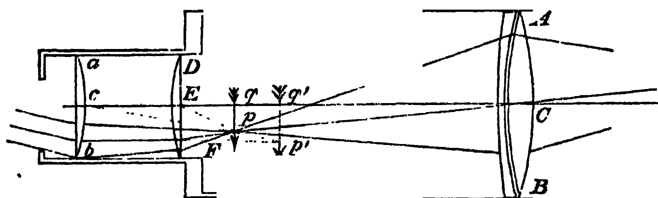
235. The position of the eye-piece is determined by the condition that a direct pencil after refraction through the field-glass may have the principal focus of the eye-glass for its geometrical focus—i.e. a direct pencil must at incidence on

DEF be converging to a point q such that after refraction through *DEF* it converge to q', the principal focus of *acb*, hence q' must be the middle point of *Ec*, and *Eq* must therefore (Art. 99) be equal to half the focal length of *DEF*, or three-fourths the distance *Ec*.

The focal lengths and position of the lenses of a Huyghens' eye-piece being determined, the *forms* of the lenses are to be chosen with reference to the use of the telescope. The conditions for diminishing *distortion*, *indistinctness*, and *curvature* of the field being different, it is a matter of judgment which of these defects are most to be avoided. According to Mr Coddington (*Optics*, q. v.) these defects will be obviated as far as possible by making the field-glass a *meniscus*, having its radii in the ratio of 11 : 4; and the eye-glass a *crossed lens*, the radii being as 1 : 6,—the forms indicated in the figure.

236. *Ramsden's Eye-piece.*

This eye-piece, sometimes called the *positive* eye-piece, is a combination of two separated convex lenses, for the purpose of diminishing the effects of spherical aberration—which can be effected better by two lenses than by one of equivalent power, because more disposable quantities are thus introduced into the calculations of the forms of the lenses.



The lenses are taken of *equal* focal length, and the distance between them is *two-thirds* of the focal length of either.

This eye-piece is not achromatic.

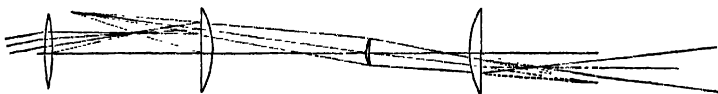
The passage of a pencil through an Astronomical Telescope provided with a Ramsden's eye-piece is indicated in the figure.

The position of the eye-piece, when in adjustment, is determined by the condition that a direct pencil at incidence on the field-glass must be diverging from a point q such that after refraction through DEF it may diverge from a point q' coincident with the principal focus of acb: hence since $Ec = \frac{2}{3}$ of focal length of acb, therefore $Eq' = \frac{1}{3}$ of the same focal length, and therefore (Art. 99) $Eq = \frac{1}{4}$ of the same.

The considerations which lead to the *forms* of the lenses are similar to those mentioned in the case of Huyghens' eye-piece. The lenses are generally of the forms in the figure—the field-glass being plano-convex, the eye-glass convexo-plane.

237. *The Erecting Eye-piece.*

The inversion of the image by an Astronomical Telescope, when furnished with either of the eye-pieces already described, renders it unsuitable for viewing terrestrial objects. To remedy this, an *Erecting Eye-piece* of four lenses is commonly used. The manner in which an object is seen through a telescope with this eye-piece will be sufficiently understood from the course of a pencil traced in the figure.



The distances, forms, and focal lengths of the lenses are adjusted to diminish as far as possible chromatic and spherical aberration.

The re-inversion of the image is sometimes effected by an eye-piece of three lenses; but in all the erecting eye-pieces that I have met with, the correction for chromatic dispersion has been very imperfect.

Note. When an object is viewed through an Astronomical Telescope furnished with an Erecting Eye-piece the parts of the image seen will appear in the same relative positions as the corresponding parts of the object:—there is no *inversion* as regards *up* and *down*, nor *reversion* as regards *right* and *left*—and the *image* of a *moving* object will *move* in the same

direction as the object itself *moves*. Such an arrangement makes the telescope very convenient for observing land objects.

Compare *note*, p. 198.

238. In the description of the eye-pieces they have been supposed to be employed in an Astronomical Telescope. The same eye-pieces are however employed with the Reflecting Telescopes. In Galileo's telescope a single eye-lens is generally used because the refraction through it is centrical. The investigations of the field of view will still be true in telescopes with compound eye-pieces, if the field-glass of the eye-piece be used in them instead of the eye-lens. The determination of the magnifying powers will also hold good, if the simple eye-lens be supposed such as will refract an excentrical pencil in parallel rays at the same inclination to the axis of the lenses as it has at emergence from the eye-glass (Art. 138) —i.e. if we substitute the *equivalent lens* for the eye-piece.

On the theory of eye-pieces, see Grunert, *Optische Untersuchungen, Theil* III., and two memoirs by Mr Airy in the *Cambridge Phil. Trans.* Vols. II. and III. Also on the eye-piece for correction of atmospheric dispersion, see *Monthly Notices*, vol. 29. p. 333, and vol. 30. p. 57.

239. Since the field of view of a telescope is of finite extent, it is necessary to have certain points in the field to which an object observed for the purpose of measurement may be referred. This is in general attained by fixing in the tube of the telescope *a set of fine parallel threads* in a plane perpendicular to the axis of the lenses, and *one or more threads at right angles to this set*, which if placed at one of the images formed by the telescope are, like that image, distinctly visible through the eye-glass. Such a set of threads are commonly called *cross-wires* or *spider lines*:—a line joining the central point of the set with the centre of the object-glass is the *line of sight* of the telescope, and with this line the optical axis of the object-glass (Art. 97) ought to coincide.

In Huyghens' Eye-piece the wires would be placed at the principal focus of the eye-glass, and therefore would be dis-

torted by excentrical refraction through that lens alone; while the image seen would be distorted by excentrical refraction through the field-glass and eye-glass, and consequently in a different degree from the wires. In this case the position of any point of the field would be estimated incorrectly by referring it to the wires, and thus Huyghens' Eye-piece can never be used in a telescope intended for measuring.

In Ramsden's Eye-piece the image given by the object-glass is formed in front of the field-glass, and at this image the wires are placed. The image and the wires are thus each seen by two excentrical refractions, and are therefore distorted in the same degree, so that the position of a point in the former is correctly estimated by referring it to the latter. This therefore is the eye-piece in telescopes used for obtaining measurements.

Galileo's Telescope can never be employed in measuring, because the image is a virtual one behind the eye-lens.

240. In an Astronomical or Galileo's Telescope, by moving the eye-glass *inwards* so as to bring it nearer to the object-glass, the pencil from any point of the image emerges from the eye-glass in a state of divergence (Art. 103), and is therefore adapted for a short-sighted eye.

In viewing a *near object* such that a pencil from any point of it cannot at incidence on the object-glass be supposed to consist of parallel rays, but has a sensible divergency, the eye-glass must be moved *outwards*.

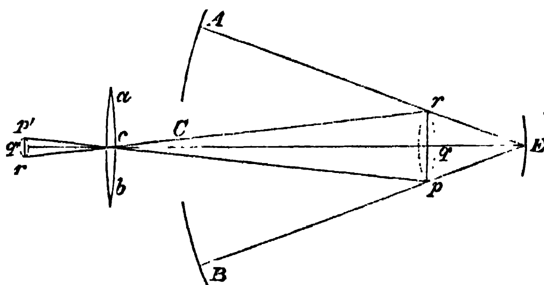
The corresponding adjustments are effected in the telescopes of Gregory and Cassegrain by moving the small mirror by a fine screw.

241. *To determine practically the magnifying power of a telescope.*

If the light of the sky fall upon the object-glass or large mirror of a refracting or reflecting telescope, a real image of that lens or mirror is formed by the eye-piece in the same manner as of a self-luminous object in the same position. The magnifying power of the telescope is approximately equal to the quotient of the diameter of the object-glass or

mirror divided by the diameter of its image thus formed. The diameter of the former can be directly determined, and the latter can be measured by a contrivance of Ramsden's, called a *Dynamometer*—whence the numerical magnifying power of the instrument is obtained.

242. The fact that the *linear* magnifying power of a telescope is equal to the ratio of the diameter of its object-glass—or large mirror—to the diameter of the bright image of the same, may be separately proved for each telescope with any given eye-piece. It may suffice for us to shew its truth in one case—and we will take Gregory's Telescope with a simple eye-lens.



Let pqr be the image of the large mirror ACB formed by the small mirror whose centre is E, very nearly at its principal focus, $p'q'r'$ the bright image of pqr, which the eye-glass whose centre is c forms, and which from the largeness of CE may be considered as the principal focus of the eye-glass. Then if F, f_1, f_2 be the focal lengths without regard to sign of the large mirror, small mirror and eye-glass severally—by triangles which are very nearly rectilinear and similar, we get

$$\frac{\text{diameter of mirror}}{pr} = \frac{CE}{Eq},$$

$$\text{and } \frac{pr}{p'r'} = \frac{cq}{cq'};$$

$$\therefore \frac{\text{diameter of mirror}}{p'r'} = \frac{CE}{Eq} \cdot \frac{cq}{cq'}$$

$$= \frac{F+f}{f} \cdot \frac{F}{f_e}, \text{ nearly,}$$

$$= \frac{F^2}{f \cdot f_e}, \text{ nearly,}$$

= magnifying power of the telescope (Art. 220).

This method is not applicable to Galileo's Telescope, because the image of the object-glass formed by the eye-lens is *virtual*.

Note. A simple and elegant proof for *any* telescope of the statement at the beginning of this Article is given by E. Hill, M.A., *Oxford, Cambridge and Dublin Messenger of Mathematics*, Vol. v. p. 84, 1871.

243. The *Dynamometer* above referred to (Art. 241), as originally used, consisted mainly of a slip of mother-of-pearl with a scale of tenths of an inch engraved on it, on which the bright spot was received so that its diameter could be read off at once. Much greater accuracy is now attained and an eye-piece with a *divided field-glass* is employed, one half of this lens remaining fixed and giving a bright circular image of the object-glass in a permanent position—the other half lens can be moved by a screw transversely to the axis of the instrument, and the bright image formed by it is brought into contact with the other fixed image—first on one side of it and then on the other: the difference of the readings of the graduated head of the screw in these two positions affords a means of determining the diameter of the bright image within a *five-hundredth* part of an inch.

244. *Microscopes.*

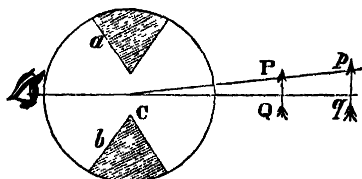
Some objects are so minute that when they are viewed by the naked eye at the least distance of distinct vision, the distances of their parts subtend no appreciable angles, and therefore cannot be discerned. In these cases it is advantageous to view an *image* of the object, instead of the object itself,—and an instrument for this purpose is called a *microscope*.

Microscopes are called *simple* or *compound*, according as a real image of an object viewed by them *is not* or *is* formed.

245. A *single lens*, or a *sphere*, forms a simple microscope.

If an object be placed nearer to a convex lens than its principal focus, an erect and magnified image of it may be seen by an eye on the axis of the lens (Art. 201).

Also if a small object PQ be placed nearer to the centre of a refracting sphere than its principal focus, a pencil diverging from a point P will after direct refraction through the sphere diverge very nearly from some point p in CP produced, and pq an erect image of PQ is thus formed. This image, if its distance from an eye close to the sphere be not less than the least distance of distinct vision, may be seen by the eye by direct pencils.

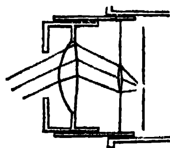


The distinctness will be much improved, if the visual pencils be restricted to pass nearly through the centre of the sphere—which can be effected by filling up a groove a, b, leaving only a small circular opening at the centre of the sphere—the effect of spherical aberration will be thus almost entirely obviated.

246. Since the minimum aberration (see Arts. 130, 130*) for parallel rays is less for substances of high refractive power, it is of great advantage to construct lenses for this purpose of substances for which μ is large;—for example, there is much less aberration in a lens of *zircon* ($\mu = 2$) than in a lens of crown-glass. The diamond has a still higher refracting power than this (see p. 179), which combined with its low dispersive power makes it the most desirable substance to be used for this purpose.

A simple microscope preferable to a single lens is composed of two convex lenses separated by a small distance on a common axis. If an object be placed nearer to the first

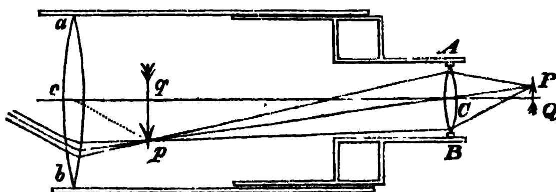
lens than its principal focus, so that a virtual image of it may be formed by each lens, the image formed by the second lens will be distinctly seen by an eye whose axis is the axis of the lenses, and whose distance from this image is not less than the least distance of distinct vision. This is the principle of Wollaston's *Microscopic Doublet.*



247. *The compound refracting microscope is an Astronomical Telescope adapted for viewing near objects.*

ACB is a convex lens called the object-glass, and *acb* a convex lens called the eye-glass, fixed in a tube whose axis is the axis of the lenses. The distance of the centres of the lenses admits of being altered for the purpose of adjustment.

If the axis of the lenses be directed to a point *Q* in an object *PQ* which is farther from the object-glass than its principal focus, the pencil from a point *P* after refraction through the object-glass converges very nearly to a point *p* in *PC* produced, and thus *pq* an inverted image of *PQ* is formed. The position of the eye-glass is such that this image is at its principal focus, and therefore the pencil from any point *p* of the image consists after excentrical refraction



through the eye-glass of rays parallel to *pc*, and suited to give distinct vision to an eye applied to the eye-glass, and thus an inverted image of the object is seen through the microscope.

248. Compound object-glasses and eye-pieces are commonly employed for reasons similar to those which render them necessary in telescopes.

For a full account of the microscopes of different makers, the mode of mounting and illuminating the object under observation, &c., and the recent development of microscopic science, see *The Microscope, its History, Construction, and Applications*, by Jabez Hogg, *tenth* edition, 1882; also *The Microscope and its Revelations*, by Dr Carpenter; *How to work with the Microscope*, by Dr Beale; also the important articles *Microscope* in the *Penny Cyclopædia* and in the *Encyc. Britann.*

249. *Def.* The magnifying power of a compound Microscope may be estimated by the ratio of the angle which the image seen subtends at the eye to the angle which the object would subtend at the eye, if placed at the distance of distinct vision and viewed directly.

Thus if K = distance of distinct vision,

$CQ = u$, f, F focal lengths of ACB, acb without reference to sign, we have

$$\text{angle subtended by the image} = \frac{pq}{cq} = \frac{pq}{F},$$

$$\text{angle which object would subtend at distance } K = \frac{PQ}{K};$$

$$\therefore m = \frac{K}{F} \cdot \frac{pq}{PQ} = \frac{K}{F} \cdot \frac{Cq}{u}.$$

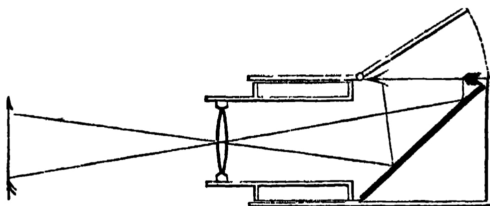
But numerically $\frac{1}{Cq} + \frac{1}{u} = \frac{1}{f}$; $\therefore Cq = \frac{fu}{u-f}$;

$$\therefore m = \frac{Kf}{F(u-f)}.$$

250. *The Camera Obscura.*

If in an aperture in the wall of a darkened room there be inserted a single convex lens, or a combination of lenses of considerable negative focal length, a real image of external objects is formed at a distance from the lens. If this image

be received on a screen, either directly or after the direction



of the pencils has been altered by reflexion at a plane mirror, an inverted picture of external objects is visible.

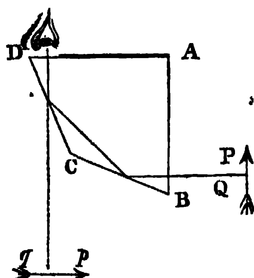
The annexed diagram represents a box constructed on the same principle, the image formed by the lens being received after reflexion at a plane mirror, on a piece of oiled paper, or ground glass, from which extraneous light is shaded by a lid which can move up and down.

251. If an object be placed before a convex lens, or combination of lenses, at a distance a little greater than that of the principal focus, and be illuminated by the sun or a powerful artificial light a real inverted and magnified image of the object is formed, and if received on a screen in a darkened room will be seen as a picture on the screen.

This is the principle of the *Solar Microscope* and the *Magic Lantern.*

252. *The Camera Lucida* of Dr Wollaston.

ABCD is a section of a quadrilateral prism of glass made by a plane perpendicular to the four planes which bound it; the $\angle A = 90°$, the $\angle C = 135°$, and $\angle B = \angle D = 67°30'$. The surface *AD* except a small portion near *D* is blackened so as not to allow the passage of light.



Let *PQ* be a luminous object placed before the side *AB*. The axis of a pencil from a point *P* of this object after passing nearly perpendicularly through *AB* is incident on *BC* at an angle exceeding

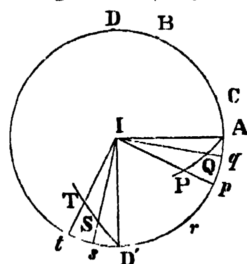
the critical angle—which between air and glass is about 41° 49′, and therefore is totally reflected; in a similar manner it is totally reflected at CD, and then emerges through AD. If pq be a screen—of paper for example—and if a pencil from a point p of it after refraction through the prism near to D emerge in the same direction with the pencil from P, then if the screen be sufficiently distant, the image of P and the point p of the screen are seen together by an eye at D, and a representation of the object PQ is visible on the screen.

This instrument is sometimes used for tracing the elevation of a building, copying designs on an altered scale, &c.

There exist several different constructions with the same object, which it is unnecessary to describe here.

253. The following investigation will shew that the picture of PQ seen on the screen is the same as the projection of PQ upon a plane parallel to AB.

Let radii of a sphere parallel to the edges of the prism, and to the normals to its four surfaces in the order according to which light falls upon them, meet the surface of the sphere in I, A, B, C, D'. Then I is the pole of the great circle $ABCD$, and



$$AC = \frac{\pi}{8} = BD, \quad BC = \frac{\pi}{4}.$$

Also let radii parallel to the axis of a pencil before and after refraction into the prism, before and after refraction out of the prism, meet the sphere in P, Q, S, T, these radii being drawn in the direction opposite to that in which the light proceeds.

Let great circles through I and these points meet the circle $ABCD$ in p, q, s, t respectively. Produce DI to D' and join IA, ID'. Draw the great circles AQP, $D'ST$, and also the great circle Ir through the direction of the axis of the pencil after one reflexion. Then

$$Bq + Br = \pi, \text{ or } Aq + Ar = \frac{\pi}{4},$$

$$Cr + Cs = \pi, \text{ or } D's + Ar = \frac{\pi}{4};$$

$$\therefore Aq = D's.$$

Also $IQ = IS$, (Art. 78); $\therefore \angle QAI = \angle SD'I$,

$$\text{and } \frac{\sin TD'}{\sin SD'} = \mu = \frac{\sin PA}{\sin QA};$$

therefore the triangles IPA, ITD' are equal in all respects;

$$\therefore IT = IP \text{.....................(i)},$$

$$\text{and } \angle TID' = \angle PIA;$$

$$\therefore \angle TIP = \frac{\pi}{2} \text{.....................(ii)}.$$

From (i) it appears that the axis of a pencil refracted and reflected by the prism of a Camera Lucida has at incidence and emergence the same inclination to any edge of the prism, and from (ii) it appears that planes parallel to the edges of the prism drawn through the axis of the pencil at incidence and emergence are perpendicular to each other. The effect of the prism therefore is merely to turn through 90°—about an axis parallel to the edge of the prism—the plane of the axis of any pencil, whilst in this plane the axis preserves the same direction as before relatively to the same edge.

Hence the picture seen on the screen pq is the same as the projection of the object PQ (fig. Art. 252) upon a plane parallel to AB.

254. *Hadley's Sextant, or Quadrant.*

AC, AB are bars of metal united at A, the centre of a

circular arc of from 50° to 70°, to which they are attached at C and B. AD another bar turning about a hinge at A, and carrying a pointer D and vernier along the arc BC. At F and A are two plane reflectors whose surfaces are perpendicular to the plane ABC: the former is fixed to AC, the latter is moveable with AD, and is parallel to F when the pointer D coincides with the point E of the arc CB. Hence in any other position the angle DAE is the inclination of the mirrors to one another. Of the mirror F the lower part only is silvered, so as to allow the passage of direct rays close to the edge of this reflecting part. G is a small telescope attached to AB, the axis of its lenses being parallel to the plane ABC and passing through the division between the silvered and unsilvered parts of F.

The instrument is used to measure the angular distance between two distant points.

Let P, Q be two points whose angular distance is required. The plane ABC being brought into the same plane with them, and the telescope pointed to Q, let AD be moved until P seen through the telescope by a pencil reflected in succession at A and F appears to coincide with Q, which is seen directly. In this case the deviation of the axis of the pencil is the angular distance of P and Q. But the deviation of the axis is double the inclination of the mirrors (Art. 77), or double the angle DAE. Hence if EC be graduated from E as the zero point, every half degree being marked as a whole one, the reading corresponding to the position of the pointer D will be the angular distance of P and Q.

Obs. The mirrors F and A are called the horizon-glass and the index-glass respectively.

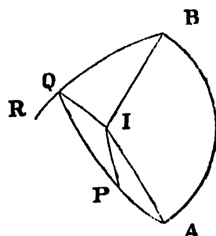
The angle measured by the instrument is the inclination of PA to QFG—if the points P, Q be distant this will coincide sensibly with the angle which they subtend at G, i.e. at the eye of the observer.

255. *Note.* If when the pointer D is at E the zero point, the planes of the mirrors, supposed perpendicular to the plane ABC, are not accurately parallel, the angular distance of two objects determined by the instrument will be affected

with a constant error called the *index-error*. This correction, which must be made to observations made by the quadrant, is equal to the *reading* of the limb when the mirrors are exactly parallel—which is the case when they are so adjusted that a very distant bright point, as a star, seen distinctly through the telescope, coincides with its image formed by reflexion at the two mirrors.

256. *To find the correction to the angular distance of two objects observed by a sextant, wherein the axis of the telescope is not exactly perpendicular to the intersection of the plane mirrors.*

In the figure constructed as in Art. (126) let IP, IR be nearly quadrants and equal to the angle between the axis of the telescope and the intersection of the plane mirrors. Let θ be the reading of the instrument, or double the inclination of the mirrors.



$PR = \theta + \delta$ the angular distance of the objects,

$$IR = \frac{\pi}{2} - \alpha = IP.$$

Now $PIR = \theta$ (Art. 126, Cor. 2);

$$\therefore \cos(\theta + \delta) = \sin^2\alpha + \cos^2\alpha\cos\theta,$$

which gives the correct angular distance of the objects.

Also α and δ being small, we have approximately

$$\delta = -\alpha^2 \tan\frac{\theta}{2},$$

the required correction to the reading of the limb.

If n be the number of seconds in the angle α, the correction in seconds $= -n^2 . \sin 1'' . \tan\frac{\theta}{2}$.

257. *The Reflecting Goniometer.*

The goniometer is an instrument for measuring the angle between two plane faces of a crystal, and consists of a circle

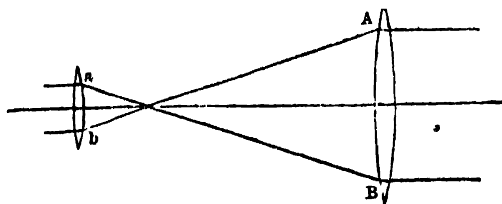
of metal turning about an axis perpendicular to its plane. The rim of the circle is graduated, and is read by a pair of verniers in opposite positions. Let the crystal be attached to the circle, so that the plane of the latter is perpendicular to the intersection of the faces of the former whose inclination is required. Bring the circle into such a position that the image of a well-defined straight line perpendicular to the plane of the circle formed by reflexion at one of the faces of the crystal coincides with another well-defined straight line which is seen directly, and read the verniers:—window-bars will answer the purpose of these straight lines very well. Turn the circle until a similar coincidence is made between the same straight line seen directly, and the image of the other formed by reflexion at the other face of the crystal, and read the verniers again. The semi-sum of the differences of the two readings of each vernier is the angle through which the circle has been turned, and is equal to the angle between the normals to the two faces of the crystal and supplemental to the inclination of the two faces.

For a full description of the construction and use of this valuable instrument, which was invented by Dr Wollaston, see the article *Crystallography* in the *Encyclop. Metrop.*

Also for a description of many optical instruments used in surveying and other practical operations see the *Treatise on Mathematical Instruments* in Weale's Series.

257*. On the *brightness* of images produced by the Astronomical Telescope, and the Intensity of their light.

Since every point of an object viewed through the Telescope must appear as a point whatever may be the magnifying power,—the *intensity* of the illumination of the several points



of the image will depend upon the quantity of light which proceeds from each point of the object and reaches the eye. The *brightness* however depends upon the impression of the whole image upon the eye. Let a beam of rays (regarded as parallel) from a point of a distant object fill the object-glass (breadth D), and emerge as a pencil of parallel rays from the eye-glass (breadth d), the breadth of the emergent pencil being δ—the whole being supposed to fall on the eye-glass,—ϖ the breadth of the pupil of the eye, m the magnifying power of the telescope—which $= \frac{D}{\delta}$ whether a single eye-lens or an eye-piece be used; and suppose for the present that ϖ is not $< \delta$ and δ not $> d$:

also let $1 : \alpha$ be the ratio in which light is diminished by absorption in its passage through all the lenses of the telescope.

Now a beam of rays at incidence on the object-glass and at emergence from the eye-glass occupies circular areas of diameter D, d respectively—hence the brightness of the image formed by the eye-glass $\propto \alpha \frac{\delta^2}{D^2} \propto \frac{\alpha}{m^2}$; and if we consider that the eye applied to the eye-glass selects a portion of each pencil which passes through the telescope, and call B the brightness of the image seen through the telescope, and I the intensity of the light in it—each being supposed to be unity for the object as viewed by the naked eye,—we shall obtain without much difficulty

$$B = \alpha \frac{D^2}{m^2 \varpi^2}, \quad I = \alpha \frac{D^2}{\varpi^2}.$$

Now so long as $m < \frac{D}{\varpi}$,—which however occurs only in telescopes of large objective apertures and low magnifying power,—the quantity B must remain constant and $= \alpha$; for if m is $< \frac{D}{\varpi}$ the diameter of the emergent pencil from the eye-glass will be greater than can be received by the pupil: the eye then receives no more of the light than if the object-glass had the diameter $m\varpi$. Hence the greatest value of B is α and can never be greater in the telescope. Since in the best achro-

matic telescopes $\alpha = 0{\cdot}85$ we see that the brightness of an object is always greatest with the naked eye. As soon as m is $> \frac{D}{\varpi}$, the brightness rapidly diminishes as the square of m.

On the other hand, I or the intensity of the light, is constant as soon as $m = \text{or} > \frac{D}{\varpi}$, provided that the field of view always includes the whole of the magnified object. I can therefore become very great when D is great, and this is the reason why exceedingly faint stars can be seen through a telescope with a large object-glass. The diameter ϖ of the pupil (which may be assumed to be about 0·2 of an inch) is not only different in different observers, but also varies with the absolute intensity of the light of the object viewed—*e.g.* it is less when we view the Moon, greater when we view Saturn; less when we view the Moon through a telescope of 5 inches aperture than through one of 2 inches aperture.

The *sky* or *ground of the heavens* has a certain degree of brightness not only in daytime, in twilight and moonlight, but even at night in the absence of the Moon. This brightness of the sky also diminishes in the telescope as $\alpha \frac{D^2}{m^2\varpi^2}$; and therefore the ratio of the brightness of an observed object to the brightness of the sky remains constant for all magnifying powers. This is the reason why for considerable magnifying powers we do not observe a correspondingly great decrease of brightness. But if we call this brightness of the sky b, although the ratio $B : b$ remains constant, our eye can nevertheless no longer distinguish the *difference* $(B - b)$ of the brightness of the object and the sky when this difference is very small. Hence faint nebulæ, tails of comets, &c. become invisible under high magnifying powers. The intensity of the light of the portion of the sky which we see in the telescope varies inversely as m^2 nearly. This intensity of the light of the field may be so great as wholly to prevent our seeing objects of feeble intensity. This is the reason why with the comet-seeker (a telescope of large aperture and small magnifying power) we cannot see stars, even of the first magnitude,

in the daytime, when we can see them without difficulty with telescopes of much smaller aperture and greater magnifying power. This also explains why with high magnifying powers we often discover very faint stars which are wholly invisible in the same telescope with lower powers.

The more perfect a telescope is, the more nearly will the image of a star resemble a bright point; and according to the above, we may without hesitation always employ for the observation of fixed stars the highest magnifying powers.

The substance of this article is the explanation of the working of a telescope given by *Olben.*

See Chauvenet's *Astronomy*, Vol. II. p. 16.

Consult also Deschanel's *Natural Philosophy*, Prof. J. D. Everett's Edition, 1882, pp. 1050—1056,—on *Measure of Brightness* and *Effective Brightness* and *Intrinsic Brightness.*

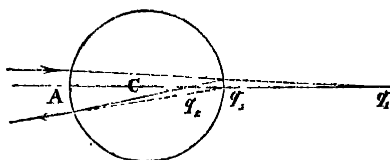
CHAPTER XI.

OF THE RAINBOW, ETC.

258. IF a pencil of light be refracted into a sphere, when it is incident on the interior surface of the sphere, a portion of it emerges and another portion is internally reflected; this latter portion being again incident on the interior surface is partially reflected and partially refracted; and so on continually.

The intensity of light in the pencils which thus successively emerge rapidly decreases.

259. Let C be the centre of a sphere of water, the re-

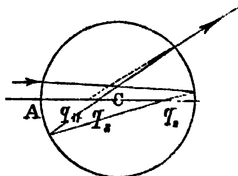


fractive index of which out of air for rays of mean refrangibility is 1·335, or $\frac{4}{3}$ nearly. Let a pencil of parallel rays of homogeneous light whose axis is in direction AC be incident directly on the sphere.

Take $Cq_1 = 3AC$, $Cq_2 = \frac{3}{5}AC$, $Cq_3 = AC$, (Arts. 29, 24), then $q_1 q_2 q_3$ are the geometrical foci of the pencil when refracted into the sphere, reflected at the internal surface and emergent from the sphere respectively.

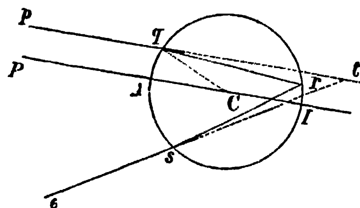
Again, if we take

$$Cq_1 = 3AC,\ Cq_2 = \frac{3}{5}AC,\ Cq_3 = \frac{3}{11}AC,\ Cq_4 = \frac{2}{5}AC,$$



the points $q_1 q_2 q_3 q_4$ being respectively on the side of C indicated in the figure. Then $q_1 q_2 q_3 q_4$ are the geometrical foci of the pencil, when first refracted, once reflected, twice reflected, and emergent after two internal reflexions respectively.

260. Suppose a system of parallel rays incident on a



refracting sphere—as a rain-drop—and emergent after one internal reflexion.

Let *pqrse* be the course of one of these rays incident parallel to the diameter $PACI$, and passing in a plane which contains C the centre of the sphere, and let us for the present confine our attention to rays which pass in this plane.

Let $\phi\,\phi'$ be the angles of incidence and refraction at q, $\theta =$ arc Ir, i.e. the angle which Ir subtends at C, $\psi = As$; produce es backward to meet pq produced in t, then $\angle\, etv = D =$ deviation of the ray pq.

Now the deviation at each of the refractions at q and s is $\phi - \phi'$, and at the reflexion at r is $\pi - 2\phi'$, and these deviations are all in the same direction;

$$\therefore\ D = 2(\phi - \phi') + \pi - 2\phi' = \pi + 2(\phi - 2\phi') \quad \text{(i)}.$$

In order to examine whether D admits of a maximum or minimum for different values of ϕ, we have,

remembering that $\sin\phi = \mu \sin\phi'$, and $\therefore \dfrac{d\phi'}{d\phi} = \dfrac{\cos\phi}{\mu\cos\phi'}$,

$$\frac{dD}{d\phi} = 2\left(1 - 2\frac{d\phi'}{d\phi}\right) = 2\left(1 - \frac{2\cos\phi}{\mu\cos\phi'}\right.$$

$$\text{and } \frac{d^2D}{d\phi^2} = \frac{4\sin\phi}{\mu\cos\phi'}\left\{1 - \left(\frac{\cos\phi}{\mu\cos\phi'}\right)^2\right\}.$$

Let ϕ_1 be the value of ϕ which makes $\dfrac{dD}{d\phi} = 0$,

$$\text{then } \quad 2\cos\phi_1 = \mu\cos\phi_1',$$

$$\text{and } \quad \sin\phi_1 = \mu\sin\phi_1';$$

$$\text{whence } \quad \sin\phi_1 = \sqrt{\left(\frac{4-\mu^2}{3}\right)}.$$

Now $\cos\phi'$ is $> \cos\phi$, and $\therefore > \dfrac{\cos\phi}{\mu}$;

$$\text{and } \therefore \frac{d^2D}{d\phi^2} \text{ is positive};$$

i.e. D is a minimum when $\phi = \phi_1$, and there is only this one value of ϕ for rays incident on the same side of PA which makes D a minimum.

Hence, considering rays incident parallel to PA, as ϕ increases from 0 up to $\dfrac{\pi}{2}$, the corresponding deviation diminishes from π, when $\phi = 0$

$$\text{to } \pi - 2(2\phi_1' - \phi_1) \text{ when } \phi_1 = \sin^{-1}\sqrt{\left(\frac{4-\mu^2}{3}\right)},$$

and afterwards increases from this value up to $2\pi - 4\sin^{-1}\dfrac{1}{\mu}$, as ϕ increases from ϕ_1 up to $\dfrac{\pi}{2}$.

Further, $\theta = \angle Icr = 2\phi' - \phi = \dfrac{\pi - D}{2}$.

Hence θ increases as D diminishes, and *vice versâ*, and is a *maximum* when D is a *minimum*.

Again, if v_1 be the distance from q at which two consecutive rays after refraction into the sphere intersect, we have from the formula,

$$\frac{\mu\cos^2\phi'}{v_1} - \frac{\cos^2\phi}{u} = \frac{\mu\cos\phi' - \cos\phi}{r}, \text{ since } u = \infty,$$

$$v_1 = r\,.\,\frac{\mu\cos^2\phi'}{\mu\cos\phi' - \cos\phi},$$

and $$qr = 2r\cos\phi';$$

$\therefore$ consecutive rays intersect within or without the sphere,

i. e. v_1 is $<$ or $>$ $2r\cos\phi'$—according as

$$\mu\cos\phi' <> 2(\mu\cos\phi' - \cos\phi),$$

or $$\mu\cos\phi' >< 2\cos\phi;$$

i. e. according as ϕ is $>< \phi_1$.

If $\phi = \phi_1$, then $v_1 = 2r\cos\phi'$, and the primary focus for consecutive refracted rays is on the surface of the sphere for those rays which pass with minimum deviation.

Lastly, $\psi = As = 4\phi' - \phi = \pi + \phi - D$, by (i).

Now $\dfrac{d\psi}{d\phi} = 1 - \dfrac{dD}{d\phi}$, and $\dfrac{d^2\psi}{d\phi^2} = -\dfrac{d^2D}{d\phi^2}$;

if $\phi_{,,}$ be the value of ϕ which makes $\dfrac{d\psi}{d\phi} = 0$, we obtain,

$$\mu\cos\phi'_{,,} = 4\cos\phi_{,,},$$

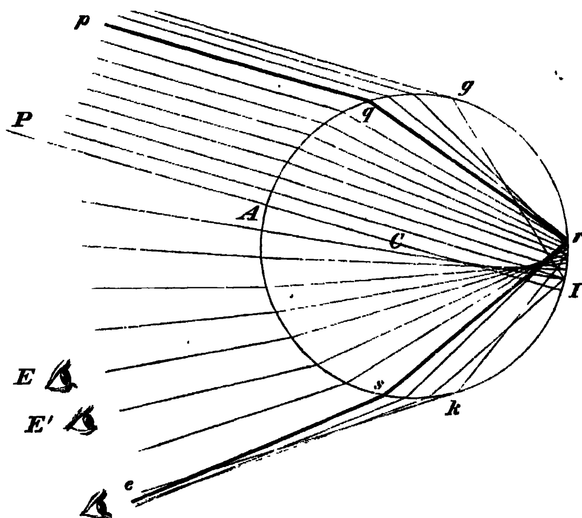
whence $$\sin\phi_{,,} = \sqrt{\left(\frac{16 - \mu^2}{15}\right)},$$

and this value of ϕ renders $\frac{d^2\psi}{d\phi^2}$ negative, or ψ a *maximum*; hence as ϕ increases from 0 up to $\phi_{,,}$, ψ increases up to a maximum and then diminishes as ϕ goes on increasing from $\phi_{,,}$ up to $\frac{\pi}{2}$.

It is easily seen that $\phi_{,,}$ is $> \phi_1$.

261. From the preceding discussion we draw the following inferences.

If *pqrse* be the ray which passes with minimum deviation, then a *small* pencil incident at q has r for its primary focus



after refraction, and emerges at s as a pencil of parallel rays in some direction *se*.

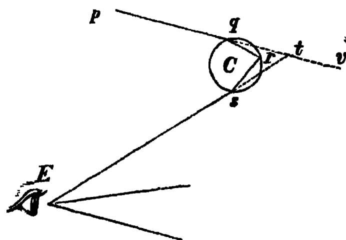
As ϕ increases continuously from 0 up to ϕ_1, ψ increases and D diminishes continuously, i. e. the rays incident on the arc Aq emerge from the arc As in a state of *divergence*.

If fg be the extreme incident ray $\left(\text{for which } \phi = \frac{\pi}{2}\right)$, and we take the arc AsK equal to the maximum value of ψ, then the rays incident along qg after refraction into the sphere are incident on its surface within the arc Ir and emerge through the arc sK: and since the deviation of the ray which emerges at s is a *minimum*, the rays which emerge from sK will by their consecutive intersection after emergence form a caustic curve, of which se is the asymptote,—the ray emergent from the extreme point K being that which was incident at $\angle \phi_{\prime\prime}$. The line se will also be an asymptote to the *virtual* caustic formed by the rays emergent along As, produced backward.

If we now suppose the whole figure to revolve about PAC as an axis we arrive at the case of a beam of parallel rays incident on the sphere, and the surface traced out by se will be a conical asymptote to the caustic surface formed by the emergent rays. If the figure represent a plane section through the axis of the beam PCI and an eye E',—if the eye be situated within the space included by AP and se, it will receive a small pencil of rays, transmitted through the sphere in the manner supposed,—the divergence of the visual pencil being less and less as the angular distance from PA is greater and greater, till when the eye is so situated that se meets it, the visual pencil consists of parallel rays, and the impression received is more vivid than for any other position of the eye: if the eye be beyond the space included by PA and se it receives no light transmitted in the manner we are considering.

262. We proceed *to explain the formation of a Primary Rainbow.*

Let C be the centre of a small spherical drop of rain falling in the air, and let a beam of sun-light fall upon it, which from the distance of the sun regarded as a point may be considered to consist of parallel rays. Suppose light homogeneous,—then of this beam a small pencil having pq for



its axis after being refracted into the sphere and once internally reflected, may emerge in a divergent state in the direction tE (Art. 261) and fill the pupil of an eye E, creating the sensation of a bright point in the sky in the direction Et of the species of light which is considered.

Through E draw EB parallel to pq, then since the deviation of pq—and consequently the angle tEB—depends merely on the angle of incidence of pq, and since all rays from the sun may be considered parallel, therefore all drops of rain whose centres lie in a conical surface of which EB is the axis and tEB the semi-vertical angle, will transmit to the eye a similar pencil of divergent rays. A drop of rain at less angular distance from EB than C will transmit to the eye a small pencil with greater deviation and greater divergence than tE, and therefore producing the sensation of a less bright point in the sky,—and the divergence of pencils which reach the eye at a greater angular distance from EB than A is less than that of the pencil tE, until when the deviation is a minimum, the pencil is nearly one of parallel rays, and the impression produced by it on the eye is greater than that of any other pencil.

The impression therefore to the eye produced by pencils transmitted in the way supposed from the drops of rain in a distant shower,—the observer being between the sun and the shower,—would be an illuminated sky of the colour of light, which has been considered, the brightness being greater at greater distances from the line EB, until it is bounded by a circle whose centre is in EB,—beyond which there is no colour, and comparative darkness.

For each species of sun-light this would be the appearance, the bounding circles for different species differing slightly in position, since the amount of minimum deviation depends upon the refractive index of the light (Art. 260). The result of superposing these illuminations of different colours will be *white light within a certain distance of the line EB terminated by a narrow circular band of vivid prismatic colours arranged in concentric circles about the line, EB.* Beyond this band there is no illumination of the sky by pencils transmitted to the eye in the manner now considered, and it will appear comparatively dark.

Further, considering the finite extent of the sun, the bands of colour such as have been described resulting from pencils proceeding from the several points of the sun, will overlap each other,—the breadth of the band will be increased by the sun's apparent angular diameter, and the colours will be more or less mixed; but the *extreme* colours of the resulting band will be unchanged.

This phenomenon produced by pencils which have been once internally reflected in the rain-drops is called the *Primary Rainbow.*

262*. *Note.* The question may suggest itself, whether two observers or two eyes—even if very near each other—see the *same* rainbow. The pencils of rays transmitted to *two* eyes with *minimum deviation* cannot be *identically* one and the same pencil:—so that the rainbows seen by two eyes are not in identically the same position in space:—even if the two eyes are those of the same observer.

Again, a rainbow seen apparently reflected in water is not an *optical image* of the rainbow seen in the sky *directly*:—but is a reflected image of the rainbow which would be seen by an eye below the water vertically below the eye of the observer,—and at a distance below the surface of the water equal to the height of his eye above it.

263. Some writers suppose that the pencils received by the eye in a state of divergence,—those namely which emerge through the arc As in Art. 261—are too faint to produce any sensible impression on the eye, and that only those pencils which pass with minimum deviation, and so enter the eye in a state of parallelism, give a sufficiently vivid sensation. This supposition affords the same explanation, as above, of the rings of prismatic colours which constitute the rainbow, but it leaves out of consideration as insensible the faint illumination of which in the previous article we have supposed the bow to be the boundary—which being of a neutral tint is not very striking.

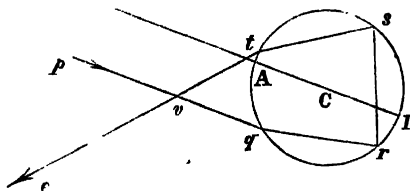
Since, however, no rays can reach the eye from drops *outside* the bow, except such as have been reflected from their

surfaces, whilst from drops *within* the bow the eye receives rays which have been internally reflected as well as rays reflected from the external surfaces of the drops, we have an explanation why the space within the bow must appear brighter than the space without it.

The fact that the rain-drops are in falling motion does not interfere with the phenomena,—the place of a falling drop being immediately supplied by another yielding the same kind of pencil of transmitted rays.

264. *Again*, suppose a system of parallel rays incident on a refracting sphere, as a rain-drop, and emergent after *two* internal reflexions.

Let *pqrste* be the course of one of these rays incident parallel to the diameter $PACI$, and passing in a plane which contains C the centre of the sphere, considering only rays which pass in this plane.



Let ϕ, ϕ' be the angles of incidence and refraction at q,

$$\psi = \text{arc } IqAt, \quad \theta = Ir, \quad \omega = Is,$$

then exterior angle $qve = D =$ deviation of the ray pq, and we shall have

$$D = 2(\phi - \phi') + 2(\pi - 2\phi') = 2\pi + 2(\phi - 3\phi');$$

$$\therefore \frac{dD}{d\phi} = 2\left(1 - 3\frac{\cos\phi}{\mu\cos\phi'}\right),$$

$$\frac{d^2D}{d\phi^2} = \frac{6\sin\phi}{\mu\cos\phi'}\left\{1 - \left(\frac{\cos\phi}{\mu\cos\phi'}\right)^2\right\}.$$

If ϕ_1 be the value of ϕ which makes $\frac{dD}{d\phi} = 0$,

$$\text{then } 3 \cos \phi_1 = \mu \cos \phi_1',$$
$$\text{and } \sin \phi_1 = \mu \sin \phi_1',$$
$$\text{whence } \sin \phi_1 = \sqrt{\left(\frac{9 - \mu^2}{8}\right)};$$

this value of ϕ renders $\frac{d^2D}{d\phi^2}$ positive, and therefore D a *minimum*, and there is only this one value of ϕ, for rays incident on the same side of PA, which makes D a minimum.

Hence considering rays incident parallel to PA, as ϕ increases from 0 up to ϕ_1 the corresponding deviation diminishes from π,—its value when $\phi = 0$,—down to $2\pi - 2(3\phi_1' - \phi_1)$, which is its value when $\phi_1 = \sin^{-1}\sqrt{\left(\frac{9 - \mu^2}{8}\right)}$, and afterwards increases from this value up to $3\pi - 6 \sin^{-1}\frac{1}{\mu}$, as ϕ increases from ϕ_1 up to $\frac{\pi}{2}$.

Further $\psi = \text{arc } IqAt = 6\phi' - \phi = 2\pi + \phi - D$.

Now $\frac{d\psi}{d\phi} = 1 - \frac{dD}{d\phi}$, $\frac{d^2\psi}{d\phi^2} = -\frac{d^2D}{d\phi^2}$.

If $\phi_{,,}$ be the value of ϕ which makes $\frac{d\psi}{d\phi} = 0$, we obtain

$$\mu \cos \phi_{,,}' = 6 \cos \phi_{,,},$$
$$\text{whence } \sin \phi_{,,} = \sqrt{\left(\frac{36 - \mu^2}{35}\right)}$$

and this value of ϕ renders $\frac{d^2\psi}{d\phi^2}$ negative, or ψ a *maximum*.

Hence as ϕ increases from 0 up to $\phi_{,,}$, ψ increases from 0 up to a maximum, and then diminishes as ϕ goes on increasing from $\phi_{,,}$ up to $\frac{\pi}{2}$.

It is easily shewn that $\phi_{,,}$ is $> \phi_1$.

265. If *pqrste* be the ray which passes with minimum deviation, it follows that the change of deviation for the consecutive ray being inappreciable,—a small pencil of which *pq* is the axis will emerge as a pencil of parallel rays of which *te* is the axis;—and by employing the formula of Art. 68, it may be shewn without much difficulty that the primary focus after the first refraction is at a distance from q along $qr = \frac{3}{4}\,qr$,—that between the two reflexions the rays are parallel, and that after the second reflexion the primary focus is at a distance from t along $ts = \frac{3}{4}\,ts$.

266. From the preceding discussion we draw the following inferences.

If *pqrste* be the ray which passes with minimum deviation, then a *small* pencil of which *pq* is the axis, emerges as a pencil of parallel rays, of which *te* is the axis. As ϕ increases

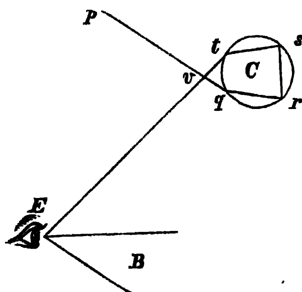
continuously from 0 up to ϕ_1, ψ increases and D diminishes continuously, i.e. the rays incident on the arc Aq emerge from the arc $IqAt$ in a state of divergence.

If fg be the extreme incident ray,—for which $\phi = \frac{\pi}{2}$,—and we take the arc $IqAtk$ equal to the maximum value of ψ, then the rays incident on the arc qg emerge through the arc tk after two internal reflexions, and these emergent rays (as in the case of Art. 261) form by their consecutive intersections a caustic curve of which te is an asymptote,—this line being also an asymptote to the *virtual* caustic formed by the rays emergent along $IqAt$, produced backward.

267. *The Secondary Rainbow.*

The space above the primary rainbow, as we have seen (Art. 263), seems darker than the rest; beyond this space appears a broader but fainter rainbow the colours of which are in reverse order to those in the primary:—the origin of this *secondary* bow can be explained by pencils which enter the eye after having been twice internally reflected in the falling rain-drops.

If $pqrstE$ be the axis of a small pencil which emerges in a state of divergence after two internal reflexions,—this may enter an eye properly situated and give the impression of a faint illumination in the direction Et. The deviation of $pqrstE$ is the *exterior* angle qvE, which is greater (and therefore the divergence of the pencil is greater) at greater angular distances from EB. By reasoning similar to that in



Art. 262, we infer that there will be a faint white light beyond a certain distance from EB bounded by a narrow circular band of vivid colours, within which there is no light received by the eye in the manner here supposed,—and which

will therefore appear comparatively dark. This phenomenon is called the *Secondary Rainbow.*

268. The theoretical existence of rainbows caused by pencils which have been three or more times internally reflected may be similarly shewn, but the rapid decrease in the intensity of light in the emergent pencils renders such rainbows with very few exceptions invisible.

In the explanation of the formation of a rainbow the sun was, in the first instance, considered a *point.* To every point of the sun's disc (which is of finite extent) there will be a corresponding rainbow, as was before noticed, and the visible rainbow resulting from the superposition of these will have its colours in some degree confused, but its general appearance will be such as has been described.

269. DEF. In any rainbow the semi-vertical angle of a cone of rain-drops which transmit to the eye pencils of parallel rays of a given colour, is the *radius of the bow* for that colour.

DEF. In any rainbow the angular elevation above the horizon of the highest rain-drop which transmits to the eye a pencil of parallel rays of a given colour is the *altitude of the bow* for that colour.

Hence the altitude of any colour in a rainbow added to the altitude of the sun's centre is equal to the radius of the bow for the same colour. If the sun's altitude exceed the radius of the bow, the bow will be below the horizon and therefore invisible.

If D_1 be the minimum deviation of light of a given kind after one internal reflexion, α the altitude of the corresponding colour in the *primary bow, A* the altitude of the sun's centre, then

$$D_1 + \alpha + A = 180^\circ.$$

If D_1 be the corresponding deviation in the *secondary bow* for the colour whose altitude therein is α, and A the same as before, then

$$D_1 - \alpha - A = 180^\circ.$$

270. *To investigate the order of the colours in the primary and secondary rainbow.*

Let the axis of a pencil of sun-light whose refractive index is μ be incident on a rain-drop at an angle ϕ and emerge after p internal reflexions. Let ϕ' be the angle of refraction corresponding to the angle of incidence ϕ, and D the deviation of the axis of the pencil.

Since $\phi - \phi'$ is the deviation produced in the axis of the pencil by each of the two refractions, and $\pi - 2\phi'$ that produced by each of the p reflexions, and these deviations are all in the same direction;

$$\therefore D = 2(\phi - \phi') + p(\pi - 2\phi') = p\pi + 2\{\phi - (p+1)\phi'\},$$

$$\text{and } \sin\phi = \mu \sin\phi'.$$

If the pencil consist at emergence of parallel rays,

$$\frac{dD}{d\phi} = 0;$$

$$\therefore 0 = 1 - (p+1)\frac{d\phi'}{d\phi},$$

and $$0 = \cos\phi - \mu\cos\phi'\frac{d\phi'}{d\phi};$$

$$\therefore \mu\cos\phi' = (p+1)\cos\phi \ldots\ldots\ldots\ldots\ldots (\text{i}).$$

Let D_1 be the deviation in this case corresponding to the limit of the colour considered. In examining its variation produced by a variation of μ, the values of ϕ and ϕ' corresponding to D_1 must be regarded as functions of μ;

$$\therefore \frac{dD_1}{d\mu} = 2\left\{\frac{d\phi}{d\mu} - (p+1)\frac{d\phi'}{d\mu}\right\},$$

$$\text{but } \cos\phi\frac{d\phi}{d\mu} = \sin\phi' + \mu\cos\phi'\frac{d\phi'}{d\mu};$$

$$\frac{dD_1}{d\mu} = \frac{2}{\cos\phi}\left[\sin\phi' + \{\mu\cos\phi' - (p+1)\cos\phi\}\frac{d\phi'}{d\mu}\right]$$

$$= \frac{2\sin\phi'}{\cos\phi}, \text{ by (i)},$$

$$= \frac{2}{\mu}\tan\phi;$$

therefore $\frac{dD_1}{d\mu}$ is *positive*, or D_1 increases as μ increases—that is, the minimum deviation of the axis of a pencil of any colour is greater as μ is greater.

The minimum deviation therefore in any rainbow is *least* for *red*, and *greatest* for *violet*.

Considering then the figure of Art. 262—in the *primary* rainbow, the *red* circle is *highest* and the *violet* is *lowest*.

In the *secondary* bow (figure, Art. 267) the *red* circle is *lowest* and the *violet* is *highest*.

These conclusions agree with the results of calculations given in the succeeding articles.

271. If we take the values of μ for extreme red and violet rays to be 1·331 and 1·344 respectively, we shall obtain,—in the *primary bow*,

for *red* $\phi_1 = 59°32'$, $\phi_1' = 40°21'$, $D_1 = 137°40'$,

$$\text{angular radius of red} = \pi - D_1 = 42°20';$$

for *violet* $\phi_1 = 58°44'$, $\phi_1' = 39°30'$, $D_1 = 139°28'$,

$$\text{angular radius of violet} = \pi - D_1 = 40°32'.$$

Hence angular breadth of the bow which would be formed by rays proceeding from one point of the sun's disc

$$= 42°20' - 40°32' = 1°48', \text{ nearly.}$$

Taking account of the pencils proceeding from different points of the sun's disc, the breadth of the bow will be increased by the sun's apparent diameter, i.e. by about 32'.

Hence angular breadth of the primary rainbow

$$= 1°48' + 32' = 2°20', \text{ nearly.}$$

It appears that the *violet* circle will be the *lowest* and the *red* the *highest* in the *primary* rainbow.

272. In the *secondary bow*,

for *red* $\phi_1 = 71°54'$, $\phi_1' = 45°34'$, $D_1 = 230°24'$,

angular radius of the red $= D_1 - \pi = 50°24'$;

for *violet* $\phi_1 = 71°29'$, $\phi_1' = 44°52'$, $D_1 = 233°46'$,

angular radius of the violet $= D_1 - \pi = 53°22'$.

Hence the total angular breadth of the bow will be

$$= 53°22' - 50°24' + \text{sun's diameter}$$

$$= 2°58' + 32' = 3°30'.$$

It appears that the *red* circle is *lowest* and the violet highest.

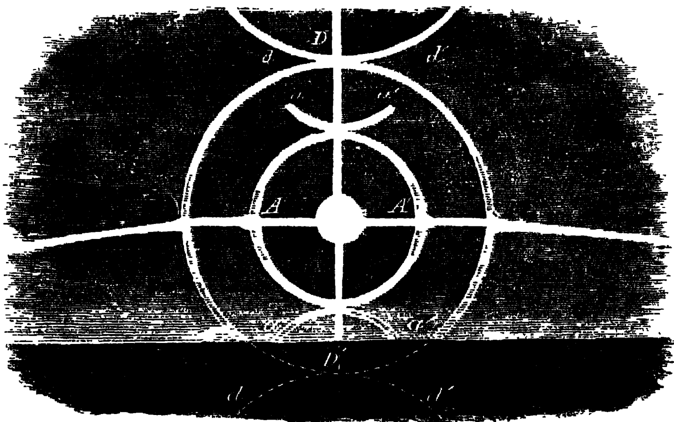
273. The investigations of this Chapter must be received as general explanations rather than as exact calculations of the phenomena of the Rainbow. A pencil of parallel rays incident on a rain-drop, and emergent after one or more internal reflexions, forms a caustic surface,—and the small oblique part of the pencil which enters the eye is determined by drawing from the eye a tangent to the caustic. Now we have assumed in the preceding explanations that the illumination is a maximum when the eye is so situated as to receive a pencil whose axis has passed with minimum deviation, but if we consider the diagram of Art. 261, it is obvious that if the eye be situated within this limiting direction *se*, as at E'',—one tangent can be drawn from the eye to the caustic formed by the rays emerging from *sk*, and another to the virtual caustic formed by the rays emerging from *sA*. And thus there may be several directions of maximum illumination, none of them coinciding strictly with the direction *es*.

The intensity of illumination of the sky in different directions has been calculated on the principles of Physical Optics, (*Cambridge Philosophical Transactions*, Vol. VI.). It is thus found that in the case of the primary bow the *principal* maximum of illumination lies a little within the position which the geometrical theory gives: also that within this there is a series of inferior maxima becoming in succession smaller. Hence

when compound light is considered there will be in addition to the principal bow a series of interior bows decreasing in brilliancy,—to these the name of *spurious* or *supernumerary* bows is given.

Two or three of the spurious bows of the primary rainbow may sometimes be seen in nature. The results of theory on this subject have been tested by measurement, by allowing a very thin column of water to run from a vessel, and receiving by a telescope artificial light which has been internally reflected in this column. (*Cambridge Philosophical Transactions*, Vol. VII.)

There is frequently observed in northern countries, and sometimes in our own climate, a regular and complex series of luminous curves surrounding the sun or moon. There is commonly discerned,



1°. a *first* circle or *halo AA'*, *red* within, *violet* without, concentric with the sun and making with him an angle of 22° or 23°;

2°. a *second* circle or *halo BB'*, similar to the preceding, placed at 46° from the sun;

3°. a diametral horizontal portion *BB'* of a very large

circle, the *parhelic circle*, on which is seen opposite to the sun a bright point, the *anthelion* ;

4°. at the points of junction of this circle with the two halos in B, B', A, A' increased intensity of luminosity, which have been taken for images of the sun;

5°. at C, C', D, D' horizontal arcs, tangents to the circular halos:—at C, C' they have little brightness,—at D, D' they are very vivid and constitute the most brilliant part of the phenomenon ;

6°. last a vertical white line DD' making a cross with BB'.

In most instances there is only seen a portion, more or less extended, of the whole phenomenon.

M. Bravais, beyond all others, has given the most complete explanation of these appearances—on the hypothesis of pencils of light transmitted to the eye after one or more internal reflexions through small hexahedral crystals of ice, which are suspended in the air by ascending currents,—especially in the cold mornings of spring and autumn.

The above picture and description is taken from M. Jamin, *Cours de Physique*, Tome III., 1869, to which the student is referred for further information. The student may also consult Dr Young's *Lectures*, or Moigno, *Optique Moderne*, Vol. I. —and the memoirs of M. Bravais on the Rainbow, Parhelia, Halos and the optical phenomena which accompany them, in the *Journal de l'Ecole Royale Polytechnique*, Tom. XVIII.

EXAMPLES AND PROBLEMS.

CHAPTER I.

Laws of Propagation of Light—direct reflexion and refraction.

1. A circular disc, six inches in diameter, is placed with its plane parallel to a vertical wall, and a person places his eye anywhere in the plane of the disc; shew that a plane circular mirror, 3 inches in diameter, will if placed in a proper position on the wall just enable the person to see the whole image of the disc.

2. A fish is floating in a cubical glass tank filled with water, with its head in one corner and its tail diagonally opposite; describe the appearance which will be presented to an eye looking towards the corner in the direction of the length of the fish, and in the same horizontal plane with it.

3. The image of a stick immersed in water is inclined to the horizon at an angle of 45^0; find the inclination of the stick, $\frac{4}{3}$ being the index of refraction from air into water.

4. The faces of two walls of a room, meeting at right angles, are covered with plane mirrors; shew that a person will be able to see but one complete image of himself in either wall.

5. A sportsman is shooting at a fish in the water; is it necessary for him to aim *above* or *below* the fish?

6. From the equation connecting the distances of the conjugate foci from the *point of incidence*, deduce the equation connecting the distances of the conjugate foci from the *centre* of the reflecting surface.

7. A pencil of parallel rays is incident directly upon a spherical refracting surface, and after refraction converges to a point at a distance from the surface equal to three times the radius—find the index of refraction (i) when the surface is concave, (ii) when it is convex.

8. If a luminous point be seen after reflexion at a plane mirror by an eye in a given position, there is a certain space within which the image of the point can never be situated, however the position of the plane mirror be changed,—find this space.

9. A speck within a solid glass cube is viewed directly through each face in succession; prove that the six images will form the angular points of an octahedron, not generally regular but having all its diagonals equal.

Also compare the volume of the octahedron with that of the cube.

10. A bright point on a glass plate is viewed by an eye close to the plate, and at a given distance from the point; find the direction in which the n^{th} image is seen after successive reflexions within the plate.

11. If the direction of a ray proceeding from a point P on the circumference of a circle, after refraction at the curve pass through a point on the circumference at an angular distance $\frac{\pi}{3}$ from P, find the point of incidence;—$\sqrt{3}$ being the refracting index.

12. A small object is placed in one focus of a prolate spheroid of glass, which is silvered at one of its poles; an eye being placed near the other pole, views the object directly, and by reflexion: if v_1, v_2 be the distances of the images seen from the eye, shew that

$$\frac{1}{v_1} + \frac{1}{v_2} = \frac{4}{\text{latus rectum}}.$$

Shew further that only one image will be seen if $\mu e > 1$,— and supposing $\mu e < 1$ compare the magnitudes of the images.

13. A luminous point is placed at the focus of a prolate spheroid, the index of refraction from which into air is equal

to its eccentricity. Find the direction after refraction of a ray falling on the further side of the spheroid. How must the direction after refraction of a ray falling on the nearer side be determined? Explain the discontinuity.

14. A luminous point is placed at one corner of a cube of some refracting substance; shew that there will be three images formed by refraction, situated at the angular points of an equilateral triangle, and that these images will be simultaneously visible, if the eye—supposed to move in a plane perpendicular to the diagonal passing through the luminous point—lie within a certain hexagon.

15. A coin is placed at the bottom of an empty hemispherical basin of given radius, and is just not visible to an eye looking over the edge—when the basin is filled with water, the whole of the coin is just visible to the eye in the same position—find the diameter of the coin.

16. A luminous point is placed in front of a refracting medium bounded by a plane transparent surface; prove that, if the bounding surface turn in any manner about a given point in its own plane,—the geometrical focus of the rays after refraction into the medium, will always be on the surface of a sphere.

17. Two rays emanate from a point in the circumference of a reflecting circle, in the plane of the circle: supposing that their n^{th} points of incidence are coincident, prove that the angle between their original directions is any one of a series of $n-1$ angles in arithmetical progression.

18. A speck is situated just within a glass sphere; shew how much of the surface of the sphere must be covered, in order that the speck may be invisible at all points outside the sphere on a line drawn from the speck through the centre.

19. If the angle of a hollow cone, polished internally, be any sub-multiple of 180°, a cylindrical pencil of rays incident parallel to the axis will, after a certain number of reflexions, be a cylindrical pencil parallel to the axis, and of the same diameter as the incident pencil.

20. A luminous point is placed within a reflecting sphere, and the light which falls directly upon the most distant point of the surface is repeatedly reflected. Shew that the distances of the geometrical foci from the centre decrease in harmonic progression.

21. An eye is placed close to the surface of a sphere of glass ($\mu = \frac{3}{2}$) which is silvered at the back; shew that the image which the eye sees of itself is three-fifths of the natural size.

22. A room has its walls covered by mirrors, shew that a man may see himself by reflexion at four of the walls, if he look in a direction parallel to either diagonal.

In what direction must he look in order to see himself after reflexion at three of the walls?

23. A circular disc exactly fits a hole in a wall and revolves in its own plane about its centre; $2n$ equidistant radii are drawn on it, and the space between every alternate pair removed: supposing the image of every object to remain the τ^{th} part of a second upon the retina of the eye, find the smallest angular velocity which may be given to the disc consistently with the eye having an unobstructed view through the hole.

If the disc itself be looked at under these circumstances, what will be the appearance presented?

24. A candle is placed at a given distance in front of a vertical plane circular mirror on a line perpendicular to the horizontal diameter at its extremity; shew that the boundary of the reflected light, which falls on a wall of which the plane is perpendicular to that of the mirror, is a parabola;—and determine its latus rectum.

25. A luminous globe falls from a point above the Earth's surface in a dark night: shew that it will look like a bright falling column, elongating as it descends.

If c_1, c_2, c_3 be the lengths of the apparent column at the ends of times t_1, t_2, t_3, from the commencement of the fall,—prove that, gravity being considered constant, and the resistance of the air being neglected,

$$t_1(c_2 - c_3) + t_2(c_3 - c_1) + t_3(c_1 - c_2) = 0.$$

26. A thin plate of glass is placed parallel to a screen, and there is a luminous point in any given position on the other side of it; the glass is cracked in one spot in the shape of two straight lines forming a cross, the planes of the cracks being perpendicular to that of the glass: shew that there will be two corresponding figures of a cross thrown upon the screen, the one bright and the other dark: shew also that the luminous point may move in a certain fixed line without producing motion in the figures.

27. A stick of given length is suspended vertically in a room. A candle is carried past the stick in such a way that the shadow of the stick falls partly on one wall, and partly on the ceiling,—the extremities of the shadow on each lying in two straight lines: find the locus of the candle.

28. An opaque sphere is placed upon a plane, and in the diameter passing through the point of contact a luminous point is placed,—its distance from the sphere being equal to the radius,—

Prove that the area of the shadow cast on the plane is three times that of a great circle of the sphere.

If a transparent liquid, whose refractive index is $\sqrt{3}$ be placed above the plane, so as just to cover the sphere, shew that the area of the shadow will then be reduced to twice that of the great circle.

29. If a concave mirror revolve round an axis through its centre, slightly inclined to the axis of the mirror, find the appearance on a screen placed at the farther focus, when a series of electric sparks passes in quick succession through the nearer focus in the axis of revolution.

30.* Three candles are placed in a room, and the two shorter being lighted throw shadows of the third upon the ceiling; if the directions of these shadows be produced, where will they meet?

31. If a globe be placed upon a table, the breadth of the elliptic shadow cast by a candle (considered as a luminous point) will be independent of the position of the globe.

32. The centre of a spherical ball is moveable in the vertical plane which is equidistant from two candles on a table: find its locus when the two shadows on the ceiling of the room are always in contact.

* * * * * * * * *

33. In Art. 24—if Q, q be conjugate foci,—and on any chord AB of the reflector a point C be taken, and QC, qC cut OB produced if necessary in R, S; then R, S will be conjugate foci.

34. The sides of a triangle are reflective, and a luminous point is placed within it: prove that the area of the triangle formed by the images varies as the rectangle contained by the segments of any chord of the circumscribing circle passing through the luminous point.

35. If a luminous point move between the centre and surface of a refracting sphere, prove that the distance between the point and its geometrical focus will be greatest when the distance of one from the centre is equal to the distance of the other from the surface.

36. A person whose height is h, observes vertically beneath his eye, an object at the bottom of a clear pool: he then removes to a distance d, keeping his eye on the object, when his line of vision makes 45^0 with the surface; shew that if $\mu^2 = 2{\cdot}5$, the depth of the pool $= 2(d-h)$.

37. A pencil of rays is directly refracted through a hemisphere of glass (radius $= r$) $p =$ the distance of origin of light from the incident surface, shew that the position of the geometrical focus will be unaltered when the hemisphere is reversed if

$$r^2 + (\mu + 1)pr - \mu(\mu - 1)p^2 = 0.$$

38. AB is the diameter of a polished semicircular arc APB. A ray of light proceeds from a point Q in the tangent at A, and after reflexion at P and B returns to Q. If the length of the ray's path be 2 feet, the mirror's diameter is very nearly 7·35 inches.

39. P is a point in a diameter AB of a sphere. If P be the origin of a pencil of rays and u, v the distances of the geometrical foci after direct reflexion at A and B, shew that

$$4uv + 4r^2 - 2ur = 3(u+v)r.$$

40. In Art. 33, if the luminous point, together with its images, form a regular pentagon, shew that the mirrors must be inclined to one another at an angle 72°.

41. A cube of glass stands in the sunshine. Prove that in general light emerges from it in twenty-eight directions.

42. In Article 33, if the angle between the mirrors be 80°, determine the positions of the point for which there will be 5 images, and those for which there will be 4.

43. A transparent sphere is silvered at the back, prove that the distance between the images of a speck within it, formed (i) by one direct refraction, (ii) by one direct reflexion and one direct refraction, is $= \dfrac{2\mu ac(a-c)}{(a+c-\mu c)(\mu c+a-3c)}$, a being the radius of the sphere, and c the distance of the speck from the centre, measured towards the silvered side.

44. In one side of a triangle, the interior of which is capable of reflecting light, there are two holes; determine by a construction the position of a point from which rays may enter so as each to pass out after one reflexion.

45. A bright point is moved about an ellipse whose focus coincides with the centre of a reflecting sphere, of such a magnitude that the directrix corresponding to the focus meets the sphere. Prove that the positions for which the geometrical focus of the point is a point in the same ellipse, are those in

which the ellipse is cut by the diameters of the sphere which are drawn to the points where the directrix meets it.

46. A luminous point is situated in a plane which cuts at right angles the line of intersection of two plane reflectors in the point O; shew that if the straight line joining the two images of the point, touch a circle the centre of which is O, the locus of the luminous point is also a circle.

47. A man 6 feet high stands in front of a looking-glass which rests on the ground and leans at an $\angle 30^\circ$ against a wall, from which he is 10 feet distant. What must be the length of the glass that he may just see his whole person?

48. Two rays are incident upon a spherical reflector in the same principal plane, and the angle between their directions before incidence is ϕ, and after reflexion is ϕ': also the angle which the arc joining their points of incidence subtends at the centre of the sphere is α, prove that $\phi + \phi' = 2\alpha$.

49. A polished wire is bent into the form of an equilateral triangle, and a luminous point moves along the sides: shew that its image after two reflexions moves along the sides of another equilateral triangle.

50. A person looking at himself in a plane mirror closes his right eye, and places his finger on the mirror so as to hide the closed eye; if he then opens the right eye and closes the left, shew that his finger will again hide the closed eye.

51. A cylindrical pencil of light is incident upon a refracting prolate spheroid in a direction parallel to the axis, the eccentricity of the spheroid being e and the index of refraction μ; find the position of the rays which emerge parallel to the axis, supposing $\mu > \frac{1}{e^2}$; and shew that none of the emergent rays will be parallel to the axis if $\mu < \frac{1}{e^2}$.

52. A ray is refracted through a sphere, its shortest distance from the centre of the sphere being $\frac{1}{n}^{\text{th}}$ the radius—

shew that if n be large the total deviation of the ray

$$= 2(\mu - 1)$$

53. If l, m, n be the direction cosines of the normal to a mirror, (λ, μ, ν), (λ', μ', ν') those of the incident and reflected ray, then

$$\frac{\lambda + \lambda'}{l} = \frac{\mu + \mu'}{m} = \frac{\nu + \nu'}{n} = 2(l\lambda + m\mu + n\nu).$$

54. The inner surface of a hollow square is polished, and a luminous point is placed at the intersection of the diagonals; shew that the number of distinct images formed after n reflexions is the sum of the series

$$4(1 + 2 + 3 + \ldots + n).$$

55. Q is a luminous point situated on AB the diameter $(2r)$ of a hollow sphere (centre O) polished at A and B: f and f' are the foci of the pencils reflected at A and B respectively —shew that

$$ff' = \frac{\frac{4}{r}}{\frac{1}{OQ^2} - \frac{4}{r^2}}.$$

56. Explain the principle of the Kaleidoscope:—and shew that when the mirrors are inclined at $60°$, the area of the field of view is to the area of the transverse section of the tube as $2\pi + 3\sqrt{3} : \pi$.

57. The locus of the image of a luminous point reflected in a plane mirror is a circle, prove that the mirror always touches a conic section.

58. The radii of the centre and inner surfaces of a spherical refracting shell are R, r respectively. Prove that the distance from the centre of the geometrical focus of a pencil of parallel rays after refraction through the whole shell is

$$\frac{1}{2}\frac{\mu}{\mu - 1} \cdot \frac{Rr}{R - r}.$$

59. A ray emanates from a luminous point P and after two reflexions at a reflecting circle returns to the point again, shew that

$$\tan^2\theta = \frac{p-q}{p+q}$$

where θ is the requisite angle of incidence, p the distance of P from the centre, and q the distance of the centre from the point where the ray in its course crosses the diameter on which P lies.

60. In the case of two plane mirrors (Art. 33) if δ lie between $\frac{\pi}{n}$ and $\frac{\pi}{n+1}$, prove that there will be $2n+1$ images when the angular distance of the luminous point from the nearer mirror is less than the lesser of the two quantities $\pi - n\delta$, $(n+1)\delta - \pi$; and that when it lies between these two quantities there will be $2n$ or $2(n+1)$ images according as the former or the latter is the lesser of the two.

61. If a ray of light after reflexion at each of the sides of a triangle in succession retrace its path, shew that it must proceed along the lines joining the feet of the perpendiculars drawn from the angular points to the opposite sides.

62. A luminous point is in the centre of an equilateral triangle; shew by considering the course of a ray parallel to one side, that the distance of the image from the luminous point for $2n$ reflexions is na, and for $2n+1$ reflexions is

$$a\sqrt{(n^2+n+\tfrac{1}{3})},$$

a being a side of the triangle.

63. A ray is incident on a refracting sphere; shew that after one internal reflexion it will emerge parallel and reversed, if $\cos\phi' = \frac{\mu}{2}$; and after two internal reflexions parallel and proceeding, if $\cos\phi' = \frac{1}{2}\sqrt{(\mu+1)}$, ϕ' being the internal angle of incidence.

64. A cylindrical tumbler being placed upon a table, a person is so situated that looking over the nearest point of

the edge he can just see half down the inside. When a fluid A is poured into the tumbler he can, by refraction, just see the bottom when the tumbler is three-fourths filled:—when a fluid B is poured in he cannot see the bottom till the tumbler is quite full. Supposing the refractive indices from air into A and B to be as $\sqrt{2} : 1$,—shew that the height of the tumbler is double the breadth of its base.

65. Three plane mirrors are joined together by their edges, so that their planes are at right angles to each other, and their reflecting surfaces turned inwards. Shew that a man looking into the solid angle so formed will see his face inverted; and trace the course of the pencils by which he sees the different parts of his face.

66. A rectangular box, at the bottom of which is a plane mirror, contains an unknown quantity of water; from the angle at which a ray of light must enter through one of two small holes in the lid in order that after refraction and reflexion it may emerge at the other,—determine the height of water in the box.

67. If a luminous point be reflected by a small plane mirror so as to be seen by an eye in a given position, and the mirror move in such a way that the luminous point always appears to be upon a given conical surface, of which the point is the vertex, and a line through the eye the axis,—find the form of the surface upon which the small mirror must always be situated.

68. A cylinder is made of a transparent substance whose refractive index is $> \sqrt{2}$; shew that when it is looked into by an eye situated anywhere on its axis produced, the whole of the inner curved surface will glisten brightly as compared with the inside of the opposite end.

69. Two plane vertical mirrors intersect at right angles and a person looks into the angle formed by them. Prove that, supposing no light can be reflected at the line of junction of the mirrors, he will see only one eye in the mirrors, and that if he shut either eye the image seen will be that of a closed eye.

70. A hollow globe of glass has a speck on its interior surface; if this be observed from a point outside the sphere on the opposite side of the centre, prove that the speck will appear nearer than it really is by a distance $\frac{\mu-1}{3\mu-1}t$, provided that t the thickness of the glass is equal to the radius of the internal cavity.

71. Explain why it is that writing paper soaked in oil becomes semi-transparent.

72. If α, β be the distances of any two geometrical foci Q, q from the surface of a spherical refracting medium, and Q', q' be any other two geometrical foci, shew that

$$\frac{\beta}{\alpha}\cdot\frac{1}{qq'}-\mu\cdot\frac{\alpha}{\beta}\cdot\frac{1}{QQ'}=\frac{\mu-1}{r}.$$

73. Two circular plane reflectors, the diameters of which are α, β, are placed so that the line joining their centres is perpendicular to the plane of each, and a bright point is placed midway between the centres; the greatest number of images of the point visible to an eye looking over the edge of the larger reflector, is expressed by the greatest integer in

$$\frac{\beta+\alpha}{2(\beta-\alpha)}.$$

CHAPTER II.

Illumination of Surfaces.

1. Two candles of unequal brightness and height are placed upon a horizontal table,—shew that the locus of the point on the table which is equally illuminated by the candles is a circle.

2. Why does the apparent brightness of a light seen by night—(neglecting such disturbing causes as absorption by the air)—remain the same at all distances?

3. A plane drawn through a given point is illuminated by two self-luminous spheres; find the position of the plane when the illumination at the given point is a maximum.

4. If a candle be placed upon a table at a given distance from a point in that table, what must be the height of the candle that the brightness at that point may be the greatest possible?

5. The shadow cast by an oblate spheroid resting on its vertex on a horizontal plane, in the sun,—is an ellipse; and the spheroid stands in its focus.

6. If the direct rays of the sun pass through a small hole and fall perpendicularly on a screen, the spot of light on the screen will be circular; but if the hole be of considerable size, the shape of the spot will be generally similar to that of the hole: explain this.

What phenomenon explicable on the same principles has been observed in the shadows of the foliage of trees during a partial eclipse?

7. A plane mirror revolves about a vertical axis in its own plane, and an eye in the same horizontal plane with a small jet of light observes its image: shew that the appearance will be the same as if the jet had crossed a field defined by a hole in a screen in the place of the mirror, with a velocity double of that which it would have had if it had moved as if attached to the mirror.

If the jet rapidly contract and expand, what will be the appearance of its image when the velocity of the mirror is very great?

* * * * * * * *

8. A plane mirror in the form of a circle whose rad. $= a$ turns very quickly about a diameter. The eye and a luminous point are at equal distances d from the centre of the face of the mirror, and lie in a plane through the centre perpendi-

cular to the diameter about which the mirror turns. Prove that the eye will see a line of light subtending an angle

$$2 \sin^{-1}\left(\frac{a \cos \alpha}{d}\right),$$

where 2α is the angle subtended at the centre of the mirror by the line joining the eye and the luminous point.

9. Two semicircular self-luminous plates are placed with their diameters upon a plane, and with their planes perpendicular to the line which joins their centres A and B; find the position of that point on the line AB where the illumination of the plane upon which the semicircles are placed is least.

10. A small plane touches a self-luminous paraboloid of revolution at its vertex and is then moved parallel to itself along the axis produced; prove that the illumination of the plane varies inversely as its distance from the focus.

11. A uniformly bright sphere is placed in a hollow paraboloid, the centre of the sphere coinciding with the focus; shew that the total illumination

$$\propto \frac{\sqrt{x}}{a+x} + \frac{1}{\sqrt{a}} \tan^{-1} \sqrt{\frac{x}{a}}$$

for a portion of the surface corresponding to a length x of the axis.

12. A circular window of radius c is made in a wall running north and south. Shew that the area of the illuminated portion of the inner floor caused by the sun's rays is $\pi c^2 \cot \alpha \sin \beta$—where α, β are the altitude and azimuth of the sun and the magnitude of the sun's disc is neglected.

13. A luminous point is situated at the centre of the base of a hollow perfectly reflecting cylinder of very small radius, and a horizontal screen is held over the cylinder at a height above its upper end which is half as great again as the height of the cylinder. Prove that a series of alternately darker and brighter rings is formed on the screen, the breadths of which are equal to the radius and diameter of the cylinder respectively.

14. A bright point is placed at the pole of a curve defined by the equation

$$\frac{1}{r} = \frac{1}{a} + \frac{1}{b}\cos^{2n+1}\theta;$$

shew that the illumination of the whole curve $\propto \frac{1}{a}$.

15. Supposing the sky on a cloudy day is as bright at every point as the moon's disc is at night, compare the light of such a day with that of a night when the moon is at full and shining: the moon's angular radius being $= 15'$.

16. A very narrow band of uniform breadth κ is bent into the shape of an elliptical hoop. If a luminous sphere of rad. a and brightness I be placed with its centre at one of the foci, shew that the total quantity of light received on the hoop is $\frac{4\pi^2 a^2 \kappa}{l} I$, where $l =$ lat. rect. of the ellipse.

17. A circular disc in which the brightness at distance r from the centre $= \beta r$, illuminates a small plane placed perpendicularly to the line joining it with the centre of the disc, and at a distance equal to the radius of the disc (a); shew that the illumination

$$= \beta a \frac{\pi}{2}\left(\frac{\pi}{2} - 1\right).$$

18. A curve is illuminated by a bright point on it: if the illumination at each point of the curve vary as r^{n-2}, find the equation of the curve. Ex. $n = \frac{1}{2}$.

19. If C be the centre of a sphere of radius r, and of intrinsic brightness B, shew that the illumination produced by the sphere at the point P of another surface is

$$\pi B \left(\frac{r}{CP}\right)^2 \cos\phi,$$

where ϕ is the inclination of CP to the normal to the illuminated surface at P.

If a circular disc be placed so that every diameter subtends an $\angle\, 2\alpha$ at the centre of the above sphere, shew that the quantity of light received by the disc is $2\pi^2 a^2 B$ versin α.

20. A bright point is placed at the centre of an ellipse; shew that the extremity of the major axis will be a point of maximum or minimum illumination according as the eccentricity is $>$ or $< \sqrt{\frac{2}{3}}$.

21. A pencil of light consists of two kinds, which are differently absorbed in passing through the same medium; in passing through a plate of a medium A of a unit of thickness the intensities of the emergent light for each unit of intensity of the incident light are α, β respectively,—and the corresponding quantities for a medium B are α', β'. Find the relative thicknesses of two plates of the two media, in order that the character of the light may not be changed by transmission through them.

* * * * *

22. Light emanating from a luminous circular disc, placed horizontally on the ceiling of a room, passes through a rectangular aperture in the floor: ascertain the form and area of the luminous patch on the floor of the room below.

Shew that neither the shape nor the area of the patch will be affected by any movement of the disc along the ceiling.

23. Suppose the interior surface of a hollow sphere to be self-luminous and non-reflecting, but to scatter a certain portion, say $\frac{1}{m}$, of the light which falls upon it, the intensity of scattered light following the law of emanation;—find an expression for the apparent brightness.

24. A narrow self-luminous rectangular lamina is placed with one end at the edge of a circular plate; the lamina is at right angles to the plate and its plane passes through the centre of the plate: find the whole illumination on the plate.

If the length of the lamina be equal to the diameter of the plate, its intrinsic brightness and breadth being given, prove that the illumination varies as the diameter of the plate.

25. A ray of light is incident upon one of two reflectors, inclined to each other at an angle $\frac{\pi}{n}$, in a direction parallel to a line which is at right angles to their intersection and bisects

the angle between them; supposing the intensity of a ray reflected at an $\angle \phi$ to be to that of the incident ray as $e \cos \phi : 1$, shew that the intensity of the ray after it has suffered n reflexions will be to that of the incident ray as $e^n : 2^{n-1}$.

26. Find the quantity of light received by an infinite plane placed at a given distance from a self-luminous sphere.

27. A pencil of parallel rays is incident parallel to the axis of a refracting prolate spheroid, so as after refraction to converge to the focus—shew that the illuminating power of any small pencil will vary as $SP^2.(SP-HP)$, where P is the point of incidence on the spheroid and S, H the foci.

28. If a surface be illuminated by a uniform bright sphere, and the illumination at any point be a function of the distance from the centre of the sphere, shew that the surface must be one of revolution.

29. A right cone, the radius of whose base is to its height as $1 : \sqrt{2}$, stands on a table and its surface is uniformly self-luminous: shew that the illumination of a point on the table at a distance from the axis of the cone equal to its height is

$$C\pi\left(\tfrac{1}{4} - \frac{1}{3\sqrt{3}}\right).$$

30. A luminous point is placed at one of the foci of a semi-elliptic arc bounded by the axis major: prove that the whole illumination of the arc varies inversely as the latus rectum.

31. A window, the height and breadth of which are respectively h and $2a$, reaches to the ground: shew that the illumination of a point of the floor at a distance x in front of the centre of the window varies as

$$\cot^{-1}\frac{x}{a} - \frac{x}{\sqrt{(h^2+x^2)}}\cot^{-1}\sqrt{\left(\frac{h^2+x^2}{a^2}\right)}.$$

32. If c_1, c_2, c_3 be the lengths of the meridian shadows of three equal vertical gnomons, on the same day, at three dif-

ferent places on the same meridian,—prove that the latitudes $\lambda_1, \lambda_2, \lambda_3$ of the places are connected together by the equation

$$c_1 \cdot \frac{(c_2 - c_3)^2}{\tan(\lambda_2 - \lambda_3)} + c_2 \cdot \frac{(c_3 - c_1)^2}{\tan(\lambda_3 - \lambda_1)} + c_3 \cdot \frac{(c_1 - c_2)^2}{\tan(\lambda_1 - \lambda_2)} = 0.$$

33. The illumination on a small plane area produced by a luminous straight line of small but sensible thickness κ placed parallel to the area at a distance d from it, is

$$I \frac{\kappa}{2d} \{\alpha + \beta + \sin(\alpha + \beta) \cos(\alpha - \beta)\};$$

where α, β are the angles of incidence of the extreme rays which fall on opposite sides of the normal to the plane area.

34. The interior surface of a sphere of radius b has a uniform intrinsic brightness B, and is non-reflecting, but scatters $\frac{1}{m}^{\text{th}}$ of the light which falls upon it, the intensity of scattered light following the law of emanation. Concentric with this sphere, and within it, is a black sphere of radius a. Prove that the illumination at any point of the bright sphere is

$$B \cdot \frac{1 + \left(1 - \frac{1}{m}\right)\left(1 - \frac{a^2}{b^2}\right)}{1 - \frac{\pi}{m}\left(1 - \frac{a^2}{b^2}\right)}.$$

35. Shew that the directions of the shadows of parallel straight rods thrown on a plane by a luminous point all meet in a point. Determine its position: and shew that it will trace out a curve of area $A \frac{\sin \beta}{\sin \alpha}$, if the luminous point describe a curve of area A on a second plane, where α, β are the angles made by the rods with the planes respectively.

36. Shew that the least shadow which can be formed on a plane by a parallelopiped of equal edges intercepting the rays of a luminous point at an infinite distance

$$= a^2 (4 \cos^2 \lambda - 1) \tan^2 \lambda;$$

and the greatest, on a plane perpendicular to the incident light

$$= 2\sqrt{3}\, a^2 \sin^2 \lambda.$$

The parallelopiped has one of its solid angles formed by three plane obtuse angles each $= 2\lambda$, and $a =$ length of an edge.

37. A and B are two luminous points whose intensities are as $n : 1$. P is a point in an ellipse of which they are the foci. Shew that the illumination of the curve at P is a maximum or a minimum, when

$$AP\left(5\frac{AP}{BP} - 1\right) = n \,.\, BP\left(5\frac{BP}{AP} - \right.$$

Determine which it is, and shew that for such points AP must be $> \frac{1}{6}$ and $< \frac{5}{6}$ *major axis*. Shew also that by increasing the value of n the above value of AP increases.

38. A luminous point is placed within a triangle; prove that the total illumination of the sides is a *minimum* when the illumination of each side is proportional to the area of the triangle which has that side for base and the luminous point for vertex.

39. A triangular prism whose nine edges are all equal, is placed with one of its rectangular faces on a horizontal table and illuminated by a sky of uniform brightness, shew that the total illumination of the inclined and vertical faces are in the ratio $2\sqrt{3} : 1$.

The foci of an ellipse are luminous points. Shew that the illumination at any point P

$$\propto \frac{AC^2 - BC^2 + CP^2}{CD^5}.$$

40. A luminous point is placed on the axis of a truncated conical shell; prove that the whole illumination of the surface of the shell varies as

$$\frac{c_2}{(c_2^2 + a_2^2)^{\frac{1}{2}}} \pm \frac{c_1}{(c_1^2 + a_1^2)^{\frac{1}{2}}},$$

where a_1, a_2 are the radii of the circular ends of the shell, and c_1, c_2 the distances of the luminous point from their planes.

41. A circular cone stands on a horizontal table. Vertically above the summit is a luminous point from which a small pencil falls on the parts of the cone close to the vertex. Shew that there will be a circular bright ring on the table, of radius $\frac{2h^2r}{h^2-r^2}$, where r is the radius of the base and h the height of the cone.

42. A regular tetrahedron stands on its base, and is exposed to a sky of uniform brightness; shew that the integral illumination of each of the upper faces is $\frac{B}{2} \cdot \frac{\pi b^2}{\sqrt{3}}$, where B is the brightness of the sky and b = edge of tetrahedron.

43. A vessel in the form of a right cylinder, polished on the inside, stands upright in the sunshine. It has an opaque cover in which there is a hole. Prove that, provided this hole be not intersected by a certain vertical plane, there will be a circular space in the base of the cylinder on which no light falls, whatever be the length of the cylinder; and that this is equally true after any amount of fluid has been poured in.

44. A polished hemisphere is placed with its base on a plane, and receives light in a direction perpendicular to the plane; prove that the illumination at any point of the plane by reflected rays $\propto \frac{\sin^3 \phi}{2 + \sin \phi}$, ϕ being the angle which the ray reflected to the point makes with the plane.

45. A bright point is placed at the focus of a reflector which is in the form of a paraboloid of revolution; prove that the illumination, from the reflected light, of any point of a plane perpendicular to the axis of the reflector varies inversely as $(y^2 + 4a^2)^2$,—where y is the distance of the point from the axis, and $4a$ is the latus rectum of the generating parabola.

46. In M. Foucault's experiment for determining the velocity of light, if the radius of the spherical mirror be $3\frac{1}{2}$ yds., and the revolving plane mirror be at the principal

focus of the lens, the focal length of which is $\frac{7}{2\pi}$ yds., and the image of the platinum wire be moved through a distance of ·009 of an inch when the mirror revolves 880 times in 1″, determine the velocity of light in miles per 1″.

CHAPTER III.

Aberrations of small direct pencils.

1. A pencil of parallel rays is incident on a convex reflecting surface—y, r being the semi-aperture and radius of the mirror,—the diameter of the least circle of aberration $=\frac{y^3}{4r^2}$.

2. A small pencil is directly refracted at a plane surface,—the radius of the least circle of aberration $=\frac{\mu^2-1}{8\mu^2}\cdot\frac{y^3}{u^2}$.

3. A pencil of parallel rays is refracted directly at a spherical surface,—the radius of the least circle of aberration $=\frac{y^3}{8\mu^2 r^2}$.

4. A mirror of given aperture and focal length, and of small curvature, has the form of a prolate spheroid;—shew that the aberration for parallel rays varies inversely as the major axis.

5. A segment of a paraboloid of revolution is cut off by a plane perpendicular to the axis so as to form a plano-convex lens: when a pencil of rays parallel to the axis is refracted through the lens, find the aberration in terms of the thickness,—which is so small that its square may be neglected.

6. If the law of refraction were assumed to be $\phi=\mu\phi'$, shew that the approximate error in finding the point where a ray incident near and parallel to the axis of a spherical refracting surface cuts the axis after refraction, would be $\frac{1}{6\mu}\cdot\frac{\mu+1}{\mu-1}\cdot\frac{y^2}{r}$,—using the ordinary notation.

7. If x be the distance from the principal focus of a spherical reflector of the centre of the least circle of aberration, for a pencil of parallel rays incident on the reflector, and y be the radius of the circle, prove that $27ry^2 = 64x^3$ approximately—where r is the radius of the reflector.

8. Prove that the aberration of a pencil of parallel rays incident directly on a spherical refracting surface, is less than it would be for a reflecting surface of the same shape, if the index of refraction be > 2.

9. The aberration of a ray which passes through a plate of glass of thickness t is $\left(1 - \frac{\cos\phi}{\sqrt{\mu^2 - \sin^2\phi}}\right) t$, ϕ being the angle of incidence: hence shew that if $\mu^2 = 2$ there is no aberration to the third order of small quantities.

10. A ray proceeding from a point on the circumference of a circle is reflected n times at the circle, prove that its point of intersection with the consecutive ray similarly reflected is at a distance from the centre $= \frac{a}{2n+1}\sqrt{1 + 4n(n+1)\sin^2\theta}$, θ being the angle which the ray before reflexion makes with the diameter.

11. A pencil of rays is refracted directly through a hemisphere, the distance of the origin from the plane surface which is that of first incidence being $\frac{r}{\mu^2}$, shew that the aberration of a ray incident at a distance y from the axis

$$= (\mu^2 - 1)\frac{\mu^4 y^2}{r},$$

r being the radius.

CHAPTER IV.

Focal lines of small oblique pencils.

1. The distance between the focal lines of a small oblique pencil after refraction at a plane surface is $= \left(\mu - \frac{1}{\mu}\right) \tan^2\phi \,.\, u$.

2. A small pencil of parallel rays enters a reflecting sphere, and after n internal reflexions emerges through the same aperture; find the angle of incidence and the position of the primary focal line at emergence.

3. An eye is placed under and close to the surface of a clear and stagnant fluid, and a vertical straight rod of given length is placed at a given distance from the eye; determine the form of the image, and the altitude of its highest point above the surface of the fluid.

4. If an eye be placed in air close to the surface of a clear stagnant fluid, shew that the apparent form of a circular arc in the fluid, whose centre coincides with the place of the eye, and whose plane is perpendicular to the surface, is defined by the equation

$$r = a\,\frac{\mu \sin^2\theta}{\mu^2 - \cos^2\theta},$$

where a = radius of circle, μ = refractive index, and the radius vector r drawn from the position of the eye makes the angle θ with the surface:—the image of any point being supposed to coincide with the primary focus.

5. A small pencil of homogeneous light is incident obliquely on a plane refracting surface; shew that if the obliquity be small, the distance of the point of incidence from the centre of the circle of least confusion will be a harmonic mean between its distances from the foci.

6. From the formula $\frac{1}{v_2} + \frac{1}{u} = \frac{2\cos\phi}{r}$ for finding the secondary focus of a pencil of light reflected obliquely at a concave mirror, deduce the formula $\frac{1}{v} + \frac{1}{u} = \frac{2}{r} + \left(\frac{1}{r} - \frac{1}{u}\right)^2 \frac{y^2}{r}$ for the aberration of a direct pencil.

7. A small pencil of rays suffers a series of reflexions within a spherical surface, prove that

$$\frac{1}{u_n} - \frac{1}{u_r} = -\frac{4}{c}(n-r),$$

where c is the length of each successive chord of the sphere described by the axis of the pencil, and u_n is the distance of the primary focus from the middle point of the chord after the n^{th} reflexion.

8. A straight rod lying at the bottom of a river is viewed by an eye at a given height above the surface of the water,—determine the form of the curve in which it is seen projected at the bottom of the water.

9. A small pencil of rays proceeding from the centre of an ellipse is reflected at the curve, determine the point of incidence that they may be in a state of parallelism after reflexion; and shew that the problem is impossible unless the eccentricity is $> \frac{1}{\sqrt{2}}$.

10. If Q be a luminous point placed in the interior of a reflecting circle, shew that the primary focus of a small pencil of rays incident at a point R will be at an infinite distance if $QR = \frac{1}{2\sqrt{3}}$. *chord bisected in* Q:—shew that this is only possible if the distance of Q from the centre is greater than half the radius.

11. On the polished inner surface of a sphere parallels of latitude are drawn; shew that, to an eye at the pole, these lines after n reflexions will appear on the surface of a sphere whose radius is to that of the reflecting sphere as

$$n + 1 : 2n + 1.$$

If meridians were drawn on the sphere, shew that after n reflexions, they would appear to an eye at the pole to be situated on the surface

$$r \sin (2n+1)\,\theta = \rho \sin (2n+2)\,\theta :$$

where r is the distance from the eye, ρ the radius of the sphere, and θ the angle between r and axis.

12. Prove the following geometrical constructions for determining the position of the primary focal line—

(i) after oblique reflexion at a spherical surface, (fig. Art. 66), Q the origin, O, A the centres of the surface and of the face of the reflector, Aq_1 the direction of the ray after reflection.

Draw OF perpendicular to QA, FG perpendicular to AO, join QG and produce it to meet Aq_1 in q_1 which will be the primary focus.

(ii) after refraction at a spherical surface, (fig. Art. 68). Q the origin, Aq_1 the refracted ray.

Draw QF, FG perpendicular to AQ, AO respectively:—join GO meeting Aq_1 in K,—draw KV, Vq_1 perpendicular to AO, Aq_1 respectively,—then q_1 is the primary focus.

(iii) Adapt the case (ii) to the case of oblique refraction at a plane surface.

13. A small pencil of rays suffers a series of reflexions within a polished spherical surface; if the rays are initially parallel, the primary focus after the n^{th} reflexion divides the chord of the surface which is the axis of the pencil in the ratio $2n-1 : 2n+1$.

14. A pencil, refracted obliquely at a plane surface, passes through a small square stop of area c^2 parallel to the plane at a distance a from the origin of light, shew that (i) the circle of

least confusion is *approximately* rectangular; (ii) its distance from the plane is a harmonic mean between the distances of the foci; (iii) its area is

$$\frac{c^2u^2}{a^2}\left(\frac{\cos^2\phi' - \cos^2\phi}{\cos^2\phi' + \cos^2\phi}\right)^2.$$

15. A rectangular hyperbola ($x^2 - y^2 = a^2$) is placed in a vertical position in water, the conjugate axis being in the surface: shew that the appearance of the hyperbola to an eye at the centre is the curve

$$\{\mu^2(x^2 + y^2) - y^2\}\{\mu^2(x^2 + y^2) - 2y^2\}^{\frac{1}{2}} = a\mu^2x^2,$$

the image being formed by primary foci; μ = refractive index from air to water.

Will the asymptotes be seen?

Caustics.

16. If Q be the focus of incidence, QA the axis of the pencil, F the principal focus of the mirror, and if the perpendicular from Q on AF and the line from Q perpendicular to QA cross AF at equal distances from F, shew that the primary and secondary foci will be at equal distances from and on opposite sides of the mirror.

17. In a spherical mirror, if the origin of light be in the principal focus, and the pencil of small obliquity, prove that $(v_1 - v_2)^2 = 4v_1v_2$ nearly.

18. Rays parallel to the axis of y are incident on the reflecting curve $y = \epsilon^x$, the catacaustic is the catenary

$$y = \tfrac{1}{2}\{\epsilon^{x+1} + \epsilon^{-(x+1)}\}.$$

19. Rays are incident on the parabola $y^2 = 4ax$ perpendicular to its axis, the equation to the catacaustic is

$$y^2 = \left(\frac{9a - x}{3a}\right)^2 \cdot \frac{ax}{3};$$

or, if the focus of the parabola is the origin, $a = r\cos^3\frac{\theta}{3}$.

Discuss the form, &c., of this curve.

20. Rays are incident on a circle from a point S within the circumference, and CO is drawn perpendicularly from the centre C on any ray SQ reflected at Q to the point P of the caustic; prove that the locus of P may be traced by means of the equation

$$\frac{1}{\sqrt{(CP^2 - CO^2)}} = \frac{1}{SO} + \frac{2}{QO}.$$

21. Rays emanating from a point S are reflected at a plane curve—if SY be the perpendicular from S on the tangent to the curve at any point P, and SY be produced to a point Q such that $QY = SY$, then will the caustic curve be the evolute of the locus of Q.

The same problem may be stated thus:

If with each point successively of the reflecting curve as centre, and its distance from the radiant point as radius, we describe a series of circles,—the envelope of all these circles will be a curve, the evolute of which will be the caustic.

22. Assuming the formula for the position of the primary focal line in reflexion at a spherical surface,—prove that if Q be the radiant point within the circle, QP the ray corresponding to an asymptote of the caustic, and P' the point in which PQ produced meets the circle, then $QP' = 3\,.\,QP$: and hence deduce a geometrical construction for the asymptotes of the caustic.

23. A radiant point being placed in front of a looking-glass, find the caustic curve formed by the rays emergent from the glass after reflexion at the quicksilver.

24. Parallel rays are incident upon a cylinder in a direction perpendicular to its axis; shew that the equation to a section of the caustic surface is

$$(\mu^2 - 1)\,y = \{(\mu^2 a)^{\frac{2}{3}} - x^{\frac{2}{3}}\}^{\frac{3}{2}} + \{a^{\frac{2}{3}} - (\mu^2 x)^{\frac{2}{3}}\}^{\frac{3}{2}},$$

the centre of the corresponding section of the cylinder, whose radius is a, being the origin, and its diameter perpendicular to the incident rays, the axis of x,—μ the refractive index.

25. Rays proceeding from a luminous point in the pole of the spiral $r = c\epsilon^{\theta \cot \alpha}$, are reflected at the curve, the catacaustic is a similar spiral. *Further*, the spiral will be its own catacaustic if $\frac{1}{2} \sec \alpha = \epsilon^{(2n\pi - \alpha)\cot \alpha}$, where n = any positive integer.

26. Rays fall on the cycloid $y = a \text{ vers}^{-1} \dfrac{x}{a} + \sqrt{(2ax - x^2)}$ parallel to the axis of x, the catacaustic is two cycloids of half the dimensions of the original given by the equation

$$\pm y \pm \frac{a\pi}{2} = \sqrt{\{(2a - x)(x - a)\}} + a \cos^{-1} \sqrt{\left(\frac{x - a}{a}\right)}.$$

27. Rays are incident on a concave spherical mirror (rad. a) parallel to the axis: the caustic will be defined by the equation

$$\{4(x^2 + y^2) - a^2\}^3 = 27a^4y^2,$$

the centre of the sphere being the origin, and the axis of x, parallel to the incident rays.

28. A pencil of rays emanates from the origin of the curve $r = a \cos^n \dfrac{\theta}{n}$, prove that the length of the caustic formed by reflexion at a loop of the curve is $4 \dfrac{n+1}{n+2} a$,—and the radius of curvature of caustic

$$= 2 \frac{n(n+1)}{(n+2)^2} a \sin \frac{\theta}{n} \cos^{n-1} \frac{\theta}{n}$$

at the point corresponding to θ.

29. The catacaustic of the curve $r = a \sec^3 \dfrac{\theta}{3}$, the pole being the radiant point, is a semi-cubical parabola.

30. If $\alpha\beta$ be the co-ordinates of a radiant point, the caustic curve formed by reflexion at the circle $x^2+y^2=c^2$ is

$$[4(\alpha^2+\beta^2)(x^2+y^2)-c^2\{(x+\alpha)^2+(y+\beta)^2\}]^3$$
$$=27c^4(\beta x-\alpha y)^2(x^2+y^2-\alpha^2-\beta^2)^2.$$

Cambridge and Dublin Math. Journal, II. pp. 128, 236.

The student may refer to several papers on the Caustics by reflexion at a circle, by Mr Holditch, in the *Quarterly Journal of Mathematics.*

CHAPTER V.

Successive reflexions and refractions—prisms—lenses.

1. Draw a line between two plane mirrors which may be the direction of a ray proceeding from a given luminous point after reflexion at either of the mirrors.

2. When a ray of light is reflected at any surface, the incident and reflected rays are equally inclined to any plane which passes through the normal at the point of incidence.

3. What is the greatest apparent zenith distance which a star can have as seen by an eye under water?

4. A pencil passes directly from air into water, through a plate of glass, find its geometrical focus, assuming

$$_A\mu_G=\frac{3}{2},\quad _A\mu_W=\frac{4}{3}.$$

5. What must be the inclination of two mirrors in order that a ray incident parallel to one of them may, after two reflexions, be parallel to the other?

6. A ray of light passes from one medium through three others bounded by parallel planes, the refractive indices taken in order being $\sqrt{3}$, $\sqrt{6}$, $\sqrt{2}$, $\sqrt{6}$;—if the angle of incidence at the first refraction be $\frac{\pi}{4}$, find the direction of the ray in each of the plates.

7. If the air near the surface of the ground be less dense than at small altitudes above it, there will be observed inverted images of distant horizontal objects.

Hence explain the *Mirage.*

8. If rays are incident upon a common looking-glass in an oblique direction from a candle,—*one* faint image is observed before the *principal* image,—and a row behind it diminishing rapidly in brightness. Explain this.

9. If a transparent plate lie between two media of unequal densities each less than that of the plate, and if a ray, passing from the denser of the external media into the plate, fall at the critical angle on the third medium,—the angle of incidence at the first surface of the plate will be the critical angle from the first on the third medium.

10. A ray passing through a point Q is incident upon a refracting plate; q is the intersection of the emergent ray, produced backwards, with the normal to the plate through Q; if the angle of incidence be $= \tan^{-1}\mu$ and t the thickness of the plate, prove that

$$Qq = \frac{\mu^2 - 1}{\mu^2} \cdot t.$$

11. A rod, inclined at any angle to a plate of glass, is seen by an eye on the opposite side of the plate; shew that the length of the image of the rod formed by geometrical foci is equal to the length of the rod.

Is the image formed by the refraction at the first surface of the same magnitude as either?

12. A glass plate of sensible thickness is in contact with the surface of still water, find how much the image of a point in the water will be elevated.

Find also the area of the upper surface of the glass which is effective in transmitting rays.

* * * * *

13. At the centre of one end of a glass cylinder a black spot is marked, shew that to an eye at the centre of the other end a number (n) of rings will be visible,—where n is the greatest integer in $\frac{b}{2a\sqrt{(\mu^2-1)}}$, b being the length of the cylinder and a its radius.

14. If n equal and uniform prisms be placed on their ends with their edges outwards, symmetrically round a point on a table, find the angle of each prism in order that a ray refracted through each of them in a principal plane may describe a regular polygon. Shew that the distance of the point of incidence of such a ray on each prism from the edge of the prism bears to the distance of each edge from the common centre, the ratio of

$$\sqrt{\mu^2 - 2\mu\cos\frac{\pi}{n} + 1} : \mu + 1.$$

15. If α be the minimum deviation for a prism of $\angle\, i$, and β that for a prism of $\angle\, 2i$, prove that

$$i = \cos^{-1}\left\{\frac{\sin\frac{\alpha}{2}}{2\sin\frac{\beta-\alpha}{4}}\right\} - \frac{\alpha+\beta}{4}.$$

16. If D be the minimum deviation for a prism whose refractive index is μ and angle ι, prove that

$$\cot\frac{\iota}{2} + \cot\frac{D}{2} = \mu\operatorname{cosec}\frac{D}{2}.$$

17. A luminous point is seen through a triangular glass prism, by means of pencils once internally reflected, trace the course of the axis of a pencil from the luminous point to the eye,—$\frac{3}{2}$ being the index of refraction from air into glass.

18. A prism is held before the eye with the edge vertical, and one face perpendicular to the axis of the eye; how much in azimuth of the horizon will the observer be prevented from seeing by the interposition of the prism?

19. A small pencil diverging from a given point is refracted through a prism in a principal plane and very near the edge—if the prism be turned about its edge, form the equations necessary for determining the curves on which the foci lie, and shew that these curves intersect.

20. Rays are incident at one point of a prism in all directions in a principal plane,—shew that if the refracting angle be $> \sin^{-1}\frac{1}{\mu}$, rays incident from that side of the normal which is towards the edge of the prism will not pass through,—and examine what rays will pass through.

21. If there be a small speck upon the middle of one of the sides of an equilateral prism, and a person place his eye close to the opposite edge in the plane drawn through the speck perpendicular to the axis of the prism: shew that he will see *two* specks, and find the apparent angular distance between them.

22. A small pencil of rays is transmitted through a prism in a principal plane—find the focus of emergent rays in the primary plane, taking account of the thickness of the prism;—and deduce from the result the expression for the place of the focus when a pencil is transmitted through a plate which is bounded by parallel plane surfaces.

23. Two equal prisms of the same refracting substance are placed in contact: shew that a ray of given species will be just incapable of transmission through the compound prism thus formed when the mutual inclination of their edges is $2\sin^{-1}\left(\frac{1}{\mu\sin\alpha}\right)$, α being the refracting angle of each prism.

24. Can objects be seen—(without internal reflexion)—through a prism whose refractive index is $\frac{3}{2}$ and refracting angle a right angle?

25. Two prisms are in contact with their edges perpendicular to one another; a ray perpendicular to the edge of

one prism being refracted through the combination emerges from the second perpendicular to its edge: determine the deviation, and shew that the problem is impossible if the angle of either prism exceed the critical angle.

26. The deviation of a ray refracted through a prism of small angle in a principal plane

$$=\left(\mu \frac{\cos\phi'}{\cos\phi} - 1\right) i, \text{ nearly,}$$

ϕ, ϕ' being the angles of incidence and refraction at the first surface. In what case will the deviation be $=(\mu-1)\, i$, nearly?

27. If a ray of light $QACS$ be refracted through a prism IKL in a principal plane, and if the vertical angle $KIL=\alpha$, $QAK=\theta$, $ACL=\phi$, and the whole deviation of the ray $=\delta$, then will

$$\tan\left(\phi-\frac{\alpha}{2}\right)\tan\frac{\alpha}{2}=\tan\left(\theta+\frac{\delta+\alpha}{2}\right)\tan\frac{\delta+\alpha}{2}.$$

28. A small pencil of rays traverses a prism in a principal plane with minimum deviation. The length of the path of the axis within the prism being c, and i the angle of the prism,—shew that the distance between the primary and secondary foci after emergence $=\dfrac{\mu^2-1}{\mu}\left(\tan\dfrac{i}{2}\right)^2 c$.

29. A ray passes through a prism in a principal plane, the deviation being equal to the angle of incidence, and each of them equal to twice the angle of the prism,—shew that the latter angle $=\cos^{-1}\sqrt{\left(\dfrac{\mu^2-1}{8}\right)}$.

30. When a ray is refracted through a prism in a principal plane, prove with the notation of Art. (91) that

$$\sin(D+i-\phi)=2\mu\sin\frac{i}{2}\cos\left(\frac{i}{2}-\phi'\right)-\sin\phi.$$

* * * * *

31. A person applies his eye to the middle of one face of an equilateral prism of glass ($\mu = 1\cdot5$) the edge of which is vertical; determine within what horizontal angular space objects must be situated in order to be visible to him, no regard being had to light internally reflected, and within what angular space the visible objects will be seen:—assuming the critical angle for glass to be $40^\circ\ 30'$, and that

$$\sin^{-1}\left(\tfrac{3}{2}\sin 19^\circ\ 30'\right) = 30^\circ \text{ nearly.}$$

32. A lens (of substance whose refractive index is μ') is placed so as just not to touch the plane surface of water (μ): light diverges from any point in the medium and forms a focus at a distance v from the lens; if the lens be now just immersed in the water, and v' be the new value of v, and f the focal length of the lens, shew that

$$\frac{1}{v} - \frac{1}{v'} = \frac{\mu - 1}{\mu' - 1} \cdot \frac{1}{f}.$$

33. From a prism of glass ($\mu = 1\cdot5$) whose refracting angle is $2\tan^{-1}\dfrac{3}{4}$ a prism is cut out; the edges of the prisms are in one plane, which are equally inclined to their bounding planes: shew that if a ray incident perpendicularly on one face of the exterior prism, emerges perpendicularly to the other, the refracting angle of the interior prism is $2\tan^{-1}\dfrac{9}{2}$.

34. A ray of light is incident upon one face of a prism ($i < 90^\circ$) perpendicularly to one face, prove that it will emerge at the opposite face if $\cot i > \cot \alpha - 1$ where α = critical angle of the substance of the prism.

Further, the angular space within which rays may be incident in order to pass through the prism is $\cos^{-1}\left(\dfrac{\sin(i-\alpha)}{\sin\alpha}\right)$.

35. The ends of a glass cylinder are worked into convex spherical surfaces whose radii are each equal to the length of

the cylinder; prove that the geometrical focus of a pencil after direct refraction through the ends of the cylinder is determined by the equation $\frac{\mu^2}{v} - \frac{1}{u} = -\frac{\mu^2 - 1}{r}$, where u and v are measured from the face nearest the origin of light and r = length of the cylinder.

36. If a convex lens be moved between an object and a screen, find the condition that a real image may be formed on the screen. Shew that in this case there will be two positions for which a real image will be formed, and if m_1 m_2 be the magnification of the images in these positions $m_1 m_2 = 1$.

37. A glass lens $\left(\mu = \frac{3}{2}\right)$ of given focal length is placed under water $\left(\mu = \frac{4}{3}\right)$; find the geometrical focus of a pencil of parallel rays directly incident upon the lens.

38. The least distance between an object and its image formed by a plano-convex lens of glass is 12 inches, find the radius of the spherical surface $\left(\mu = \frac{3}{2}\right)$.

39. A person views an object through a convex lens with both eyes; trace the pencil by which each eye sees the object.

40. A convex lens, held 12 inches from a wall, forms on the wall a distinct image of a candle; when the lens is held 6 inches from the wall, it is found that, to produce a distinct image of the candle on the wall, the distance of the candle from the lens must be doubled;—find the focal length of the lens.

41. From a prolate spheroid, formed of glass, portions are removed by planes passing through the two foci and perpendicular to the axis; if a luminous point be situated at one of the foci, and the eye placed close to the other, describe the

appearance which will be presented, so far as it is due to pencils which have suffered one internal reflexion.

42. Rays issuing from a luminous point in its axis are incident upon a thin lens. A portion of those that enter the lens is allowed to proceed at once through the second surface; a second portion however does not escape until it has been twice internally reflected; a third portion four times reflected, and so on. Shew that a row of images will be formed whose distances from the lens are in harmonic progression.

43. In the annular lens of a Lighthouse,—investigate the relation between its focal length and the radius of curvature of a section of one of its concentric rings by a plane through its axis.

44. If a ray of light be incident upon a refracting sphere and the directions of the incident and emergent rays produced meet the surface in the same point; find the angle of incidence when $\mu = \frac{3+\sqrt{5}}{2}$.

45. A sphere composed of two hemispheres of different refractive powers is placed in the path of a pencil of light in such a manner that the axis of the pencil is perpendicular to the plane of junction and passes through the centre; determine the geometrical focus of the refracted pencil.

46. Find the geometrical focus of a pencil of rays refracted through a hollow glass sphere, whose external and internal radii are r, r' respectively.

47. Shew what kind of lens will be required to enable a person to see distinctly under water.

If an eye be under water, shew how to determine the course of the ray by which any object above the water is seen.

48. A plano-convex and a plano-concave lens of equal

power are fitted together so as to form a plate;—if the lenses be slightly separated, remaining co-axial, examine the change they will produce in the divergency of a small pencil of divergent rays incident directly (i) on the negative lens, (ii) on the positive lens.

49. Two plano-convex lenses of the same size and form are placed so as to make a double-convex lens: find the focal length of this lens, the refractive indices of the two substances being μ and μ'.

50. A convex lens is moved between a candle and a vertical screen, the distance of which is greater than four times the focal length of the lens: prove that two real images of the candle will be formed on the screen:—and if the ratio of their magnitudes is m, find the focal length of the lens.

51. The focal length of a convex lens in air is 15 inches. When its lower surface is immersed in a certain fluid, a vertical pencil of parallel rays is brought to a focus at a depth of 24 inches: when the other surface is immersed, the depth of the focus is 40 inches; and when the lens is wholly immersed the focal length is 60 inches. Find the refractive indices of the lens and fluid, and the radii of the surfaces of the lens.

52. A pencil of rays is directly incident on a double concave lens the second surface of which is silvered; find the geometrical focus of the pencil after emergence, and the form of the spherical surface which, placed at the same distance as the lens, would reflect rays to the same focus.

53. If the radius of the anterior surface of a concave glass speculum of inconsiderable thickness $(c) = a$, and if the radius of the second surface $= a + \frac{13c}{9}$, the image of a distant object formed by reflexion at the first surface will coincide with the image formed by refraction at the first surface, then by reflexion at the second surface, and by refraction again at the first.

54. A pencil of rays is directly refracted through a series of lenses separated by finite intervals $a_1, a_2, \ldots a_{n-1}$, the axes being coincident. Shew that if $\frac{1}{k_1}, \frac{1}{k_2} \ldots \frac{1}{k_n}$ be the focal lengths of the lenses, the geometrical focus will be given by the equation

$$v_n = \frac{1}{k_n +} \frac{1}{a_{n-1} +} \frac{1}{k_{n-1} +} \cdots \frac{1}{k_1 +} \frac{1}{u}.$$

55. If r, s be the radii of the surfaces of a lens of thickness t, and if $t = \left(1 + \frac{1}{\mu}\right)(s - r)$, shew that a pencil will be refracted through the lens without aberration, if the origin be at a distance $(1 + \mu)\, r$ from the first surface, and that the divergency of the pencil will be unaltered by the refraction.

56. If μ be the refractive index of a sphere, r = its radius, μr = distance of the origin of light from the centre, prove that the extreme incident rays on emergence intersect a screen touching the sphere at the point opposite to the origin of light in a circle, whose radius is

$$\frac{2 - \mu}{2 + \mu} \sqrt{\frac{\mu + 1}{\mu - 1}}\, r.$$

57. If f be the focal length of a lens when the thickness is neglected, shew that when the thickness is sensible, but small, the principal focus will be very approximately at a distance f from the first surface of the lens if the ratio of the radii of the first and second surfaces be $\sqrt{\mu} - 1 : \sqrt{\mu}$.

58. A concave mirror of radius r has its centre in the centre of a convex lens, and the axes of the two coincide. If f be the focal length of the lens and if rays proceeding from a point at a distance u from the lens emerge after a second refraction as a pencil of parallel rays, prove that

$$\frac{1}{u} + \frac{2}{r} = \frac{2}{f}.$$

59. A person finds that he sees distinctly in a vertical plane at a distance of 5 in. and in a horizontal plane at a distance of 6 in.; he therefore uses an eye-glass the front surface of which is spherical and the posterior surface cylindrical, the axis of the cylinder being vertical; determine the curvatures of these surfaces in order that he may see distinctly an object placed at a distance of 8 in. from his eye, the eye-glass being supposed to be close to the eye. $\mu = 1{\cdot}5$.

60. Two convex lenses, of focal lengths a and b, are placed at a distance c; if P and Q be conjugate foci, F, G the respective positions of P and Q when Q and P are respectively at an infinite distance, prove that

$$PF \,.\, GQ = \left(\frac{ab}{a+b-c}\right)^2.$$

61. A double convex lens having radii r, s and a small finite thickness t is placed with its posterior surface in contact with a fixed ideal plane; shew that when the lens is reversed the position of the focus for parallel rays will be altered by the quantity

$$\frac{r-s}{r+s} \cdot \frac{t}{\mu}.$$

62. A lens is placed in the centre of a concave mirror, the axes being coincident, a pencil is incident directly on the lens, and after refraction is reflected at the mirror, and again refracted through the lens, prove that the last geometrical focus is the same as if the pencil had been once reflected at a mirror coincident with the image of the concave mirror formed by the lens.

63. If P and Q be two small equal objects in the axis of a lens and on opposite sides of it, the angle subtended at P by the image of Q is equal to the angle subtended at Q by the image of P.

64. In Art. 91. If ϕ, ψ be the angles of incidence and emergence, D the deviation, prove that

$$\cos(\phi-\psi)\,\{\cos i - \cos(D+i)\} = \cos i \,.\, \cos(D+i) - 1 + \mu^2 \sin^2 i.$$

65. Two convex lenses have the same axis, shew that the image of a point on the axis at a distance d from the nearest lens will be unaltered in position by reversing the combination, if the distance between the lenses be

$$d \cdot \frac{2f_1 f_2 - (f_1 + f_2)\, d}{(f_1 - d)(f_2 - d)}.$$

66. Two parallel rays are incident at an angle ϕ on one face of an isosceles prism and emerge at right angles, one of them having been reflected at the base; if i be the angle of the prism and μ the refractive index, prove that

$$\sin 2\phi = (1 - \mu^2 \sin^2 i) \sec i.$$

67. In different prisms formed of the same substance, prove that as the refracting angle increases the minimum deviation also increases.

68. If the minimum deviation for rays incident on a prism be α, the refractive index cannot be less than $\sec \frac{\alpha}{2}$, and the $\angle$ of the prism cannot be $> \pi - \alpha$.

69. A pencil of light passes through two prisms whose edges are parallel in a principal plane of each; if the pencil passes through each with minimum deviation, the angles of incidence on the first and emergence from the second prism being ϕ, ϕ_1 respectively, the two rays for which the refractive indices are μ, $\mu + \delta\mu$, and μ_1, $\mu_1 + \delta\mu_1$ will emerge parallel if

$$\frac{\delta\mu}{\mu} \tan \phi = \frac{\delta\mu_1}{\mu_1} \tan \phi_1.$$

70. The minimum deviation of a ray refracted through a prism of 60° is 90°; shew that the refractive index of the prism for that ray is 1·93 nearly.

71. A rod situated in a plane perpendicular to the edge of a prism (i) is viewed through the prism; find the incli-

nation of its image to either face of the prism: and shew that the image will be parallel to the rod, if

$$\cot\left(\alpha - \frac{i}{2}\right)\cot\left(\alpha + \frac{i}{2}\right) = \mu,$$

where α is the ∠ which the rod makes with the plane bisecting the ∠ between the faces of the prism, μ the refracting index of the prism.

72. BOA is a diameter of a sphere and a spherical portion described from centre A, with radius AO, is cut out; prove that if u be the distance from O in the line BO produced of an origin of light, and v the distance from O of the geometrical focus after refraction through the solid BO,

$$\frac{\mu^2}{v} - \frac{1}{u} = \frac{(\mu - 1)^2}{r}.$$

73. The minimum deviations at the three angles of a triangular prism of a ray of index μ are $\delta_1, \delta_2, \delta_3$; prove that

$$\mu^3 - \mu^2\left(\cos\frac{\delta_1}{2} + \cos\frac{\delta_2}{2} + \cos\frac{\delta_3}{2}\right) + \mu\left(\cos\frac{\delta_1 + \delta_2}{2} + \cos\frac{\delta_2 + \delta_3}{2} + \cos\frac{\delta_3 + \delta_1}{2}\right) - \cos\frac{\delta_1 + \delta_2 + \delta_3}{2} = 0.$$

74. Verify the following construction for finding the focus conjugate to a point P on the axis of a lens O, everything being supposed positive: produce OP to Q, making PQ equal to the focal length of the lens; on OQ as diameter describe a circle, and let it be cut in R by the circle described with Q as centre and QP as radius; draw RN perpendicular to OQ: measure OS in direction of and equal to PN: S is the required focus.

75. Two prisms with refracting angles each 45° are placed with one face of each in contact, and their edges at right angles. If the refractive index $= \sqrt{2}$, the minimum deviation of a ray which passes through the pair of prisms will be 30°.

76. Two prisms whose refracting angles are right angles and refracting indices μ, μ' are placed so that one face of

each is in contact: their edges are parallel and their refracting angles opposed. Prove that the minimum deviation of the compound prism is $\sin^{-1}(\mu'^2 - \mu^2)$.

77. If a ray pass through a lens without deviation and if its directions before incidence and after emergence cut the axis of the lens in two points q, q', the limiting value of qq' is maximum or minimum according as the thickness is equal to

$$\frac{(s-r)\sqrt{\mu}}{\sqrt{\mu} \mp 1}$$

where r, s are the radii of the surfaces of the lens, μ the refractive index and $s > r$.

78. If i = angle of a prism, $\mu = \sec \alpha$, D = minimum deviation, then

$$\tan \alpha \sin \frac{i}{2} = \sin \frac{D}{2} \sin \frac{D+2i}{2}.$$

Shew that if $i = 2 \sin^{-1} \frac{1}{\mu}$, where μ = refractive index for mean rays, then an eye placed against the prism will see a faint arch of red.

79. The section of a prism made by a principal plane is a triangle ABC. A ray falls on AB making an angle ϕ with the normal (measured *towards* the edge) and, after internal reflexion at BC, emerges from AC. Shew that with the usual notation

$$D = \phi + \psi + A, \; \phi' - \psi' = B - C.$$

80. If m be the linear magnifying power of a *thin* lens, for an object at a distance u from the first surface, shew that when the thickness t is taken into account, the magnifying power is increased by $m^2 \cdot \frac{\mu-1}{s}\left(1 + \frac{\mu-1}{r} u\right)\frac{t}{\mu}$,—in which t^2 is neglected: r, s being the radii of the two surfaces of the lens, and u the distance of the object from the first surface. See Arts. 99, 200.

81. Parallel rays fall on a thick lens and converge to B,

while if A be the origin of light, the emergent rays are parallel. Shew that if U, V be any pair of conjugate foci, then

$$AU \,.\, BV = \left\{ \frac{\mu r s}{\mu(\mu-1)(r+s)+(\mu-1)^2 t} \right.$$

where r, s are the radii of the bounding surfaces, and t the thickness of the lens.

82. A ray passes through a right-angled prism with minimum deviation, meeting the face of the prism at a distance y from the edge. Shew that the foci of the emergent pencil will be separated by a distance $\frac{y\sqrt{2}}{\mu}(\mu^2-1)$.

83. Prove that a concave lens can be constructed such that the direction of every ray of a pencil proceeding from a certain point shall after refraction at the first surface pass through the centre of the lens,—that in this case there will be no aberration at the second refraction,—and that the only effect of the lens will be to throw back the origin of light a distance $(\mu-1)t$, t being the thickness of the lens.

84. $ABCD$ is a double convex lens (axis BD) whose thickness $t = \mu$ (*sum of radii of surfaces*). P is a dark spot within the lens at the point of intersection of AC and BD; Q, R the images of P seen from two points in the axis on different sides of the lens. Then, if f be the focal length of a thin lens which has the same radii and refractive index as the lens $ABCD$,

$$QR = \frac{4f}{\mu-2} + \frac{\mu^2-3\mu+3}{(\mu-1)(\mu-2)}\,t.$$

85. The distances of the conjugate foci from the first and second focal centres of a lens are connected by the equation

$$\frac{1}{v} - \frac{1}{u} = (\mu-1)\left(\frac{1}{r} - \frac{1}{s} - \frac{\mu-1}{\mu}\cdot\frac{t}{rs}\right).$$

CHAPTER VI.

Refraction through media of varying density.

1. The index of refraction at any point of a medium bounded by a plane is $\epsilon^{\frac{x}{c}}$, where x is the distance of the point from the plane; find the equation to the path of a ray incident at an angle α on the bounding plane, and shew that it will have an asymptote perpendicular to the plane at a distance $c\alpha$ from the point of incidence.

2. The index of refraction (μ) in a medium varies from point to point, being a function of the distances x and y from two planes at right angles to each other; a ray traverses the medium in a plane perpendicular to these two planes; if $\log(\mu) = f(xy)$, prove that the curvature of the path of the ray varies as

$$f'(x) \cdot \frac{dy}{ds} - f'(y) \cdot \frac{dx}{ds}.$$

3. A ray of light which passes through two media bounded by parallel plane surfaces, will emerge parallel to its first direction, if the deviation in passing out of one medium into another under a given angle of incidence be supposed proportional to the difference of the densities of the media.

4. The index of refraction at any point of a medium $\propto$ (distance)$^{-\frac{1}{2}}$ from a given plane, prove that the path of any ray in the medium is a cycloid.

5. The refractive index at any point of a medium $= \left(\frac{a}{r}\right)^{\frac{3}{2}}$ where r is the distance of the point from a given point. If when $r = a$ the direction of a ray of light be inclined at an $\angle \frac{3}{4}\pi$ to the radius vector, shew that the path of the ray is a cardioid.

6. If the refractive index of a medium at any point be proportional to its distance from a fixed point, the path of the ray will be a rectangular hyperbola. If it be proportional to its distance from a given plane the path of the ray will be the curve

$$\frac{2x}{a} = \frac{c}{a} e^{\frac{y}{a}} + \frac{a}{c} e^{-\frac{y}{a}},$$

a and c being constants.

7. The refractive index at any point of a plate at a distance x from its first surface is given by $\mu = f(x)$; if v be the distance from the first surface corresponding to a distance x in the plate, prove that the position of the focus is given by

$$v = x - f(x) \int \frac{dx}{f'(x)},$$

v and x being estimated positive in the same direction.

Ex. If $\mu = 1 + ax$, find the change in the position of the focus in passing through a plate of thickness c.

8. A prism is of such material that the density of the plane section through the edge at an $\angle\ \theta$ to one of the faces is $\mu e^{\theta \tan \alpha}$;—if a ray be incident perpendicularly on this face at a distance a from the edge, shew that the distance r of the ray when incident at an $\angle\ \phi$ upon a plane section is given by the equation

$$\frac{a}{r} = e^{\frac{\phi \sin^2 \alpha}{2}} \left\{\frac{\cos(\phi + \alpha)}{\cos \alpha}\right\}^{\cos^2 \alpha}.$$

9. A ray of light passes through a medium of which the refractive index at any point is inversely proportional to the distance of that point from a certain plane. Prove that the path of the ray is a circular arc of which the centre is in the above-mentioned plane.

10. In Art. 122, shew that the projection of the radius of curvature of the path of the ray on the normal to the surface of equal density at any point of the path is equal to

$$\frac{\mu}{\sqrt{\left(\frac{d\mu}{dx}\right)^2 + \left(\frac{d\mu}{dy}\right)^2}}.$$

11. When a ray traverses a medium of variable density, shew that when it is normal to a stratum of equal density, there will in general be a point of inflexion in its path; point out the exceptions, and discuss the nature of the path in these exceptional cases.

12. In a medium bounded by parallel planes the index of refraction is the same at equal distances from both planes, and is $= \mu\epsilon^{-\frac{d}{a}}$, where d is the distance from the plane bisecting the whole thickness ($2a$) of the medium. If a pencil of light pass directly through it, find the distance between the foci of the incident and emergent light.

13. ABC, ABD are the faces of a triangular prism of small refracting angle (α), and the refractive index out of air at any point P in the prism $= \mu (1 + m\theta)$, where m is constant and θ is the angle between the planes ABC and ABP. Shew that if D be the deviation of a ray which is incident perpendicularly on ABC, and is refracted through the prism,

$$D = (\mu - 1)\,\alpha + \frac{\mu m \alpha^2}{2}.$$

14. A medium is bounded by the planes of x and y, the refractive index at any point being given by $\mu = \epsilon^{\frac{xy}{a^2}}$; two rays are incident on it parallel to the axes respectively, and at distances c from the origin; shew that if they intersect, it will be at an angle whose circular measure is $\frac{\pi}{2} - \frac{c^2}{a^2}$.

15. A ray of light passes through a medium bounded by parallel planes, the density of which varies in such a manner that the index of refraction at any point $= 1 + kx$, where x is the distance of the point from the plane on which the ray is first incident. The angle of incidence being α and the point of incidence the origin, shew that the path of the ray is defined by the equation

$$\cos^2\frac{\alpha}{2}\,.\,\epsilon^{\frac{ky}{\sin\alpha}} + \sin^2\frac{\alpha}{2}\,.\,\epsilon^{\frac{-ky}{\sin\alpha}} = 1 + kx.$$

16. A vessel of depth a, the top and bottom of which are horizontal planes, is filled with a transparent fluid, the refractive index of which at a depth z below the surface is $1 + \frac{z}{a}$. Two small holes being made in the top, a ray of light enters at one hole, is reflected at the bottom and emerges at the second hole; shew that the distance between the holes must not be $> 2a \log_e (2 + \sqrt{3})$.

17. Shew that if $\mu^2 - 1 = k\rho$ for any state of density of the Earth's atmosphere, the following differential equation of the trajectory of a ray of light coming from a heavenly body results independently of any *Theory of Light*, viz.:

$$d\theta = \frac{a \sin z \,.\, dr}{r\sqrt{(r^2 - a^2 \sin^2 z + kr^2 \rho)}},$$

z being the apparent zenith distance of the body at the Earth's surface, a the Earth's radius, and r, θ polar co-ordinates referred to the Earth's centre and to the radius drawn to the place of observation.

18. *ABCD* are four points, *abcd* are their respective images formed in any manner by refraction through any number of surfaces; *AB*, *CD*, *ab*, *cd* are bisected at right angles by the same straight line, shew that

$$\mu_1 \frac{AB \,.\, CD}{AC + AD} = \mu_2 \frac{ab \,.\, cd}{ac + ad},$$

where μ_1, μ_2 are the indices of refraction of the media in which *ABCD* and *abcd* lie respectively.

19. A ray is propagated through a medium of variable density in a plane with respect to which the medium is symmetrical,—if the refractive index at any point whose polar co-ordinates are r, θ be $\lambda re^{c\theta}$, and ψ be the angle which a ray passing through the point makes with the radius vector at the point, prove that $r = ae^{-\frac{2\theta + \psi}{c}}$, a being a constant.

20. Supposing the atmosphere to consist of spherical

strata of uniform density, prove that if the observed zenith distance of a star at a given station be z, and the tangent to the path of the ray which reaches the eye in passing through the stratum whose refractive index is μ and radius r, makes an angle ζ with the vertical at the given station,

$$d\zeta = -\frac{r_0}{r}\sin z\left(\frac{\mu^2}{\mu_0^2} - \frac{r_0^2}{r^2}\sin^2 z\right)^{-\frac{1}{2}}\frac{d\mu}{\mu},$$

where μ_0, r_0 are the values of μ, r at the given station.

21. A refracting medium consists of spherical surfaces which have a common tangent plane: the internal and external radii being a, b respectively. The refractive index of any surface varies inversely as the diameter of that surface. Shew that if a small pencil emanate from the centre of the interior surface and be refracted directly through the shell, the distance of the geometrical focus of the emergent pencil from the exterior surface is $b\left(1 + \tan \log_e \frac{b}{a}\right)$.

22. If the path of a ray cut at a constant angle a the surfaces of equal density in a variable medium, prove that $\mu = \mu_0 e^{\phi \tan a}$, where ϕ is the inclination of the path to a fixed line.

23. A ray traverses in one plane a medium in which the density is a function of the distance from a fixed point. Its path is an equiangular spiral about this point as pole. Prove that the path of any other ray which traverses the medium also in a plane is an equiangular spiral.

In general if the path of the first ray be known, prove that that of the second belongs to a certain family.

24. If the surfaces of equal density be symmetrical with respect to the plane of x, y—and if in going from one to the next in the plane of (x, y)—(in which plane the ray is propagated)—the index μ may be put in the form

$$\mu^2 = (x + a) f\left(y^2 \epsilon^{\frac{x}{a}}\right),$$

shew that the parabola $y^2 = 4ax$ is a possible path for the ray to travel in.

25. Prove that in a transparent medium of variable index of refraction μ, the differential equations of a ray whose length is s are

$$\frac{d}{ds}\left(\mu\frac{dx}{ds}\right) = \frac{d\mu}{dx},\ \frac{d}{ds}\left(\mu\frac{dy}{ds}\right) = \frac{d\mu}{dy},\ \frac{d}{ds}\left(\mu\frac{dz}{ds}\right) = \frac{d\mu}{dz}.$$

If $\mu = \mu_0\left(1 - \frac{y^2}{b^2}\right)^{\frac{1}{2}}$, shew that the equation of a ray in the plane of xy is $y = \sqrt{b^2 - a^2}\sin\left(\frac{x}{a} + \alpha\right)$,

where a and α are arbitrary constants.

If x and x' are the values of x at any pair of intersections of consecutive rays, shew that

$$a^3\left\{\tan\left(\frac{x}{a} + \alpha\right) - \tan\left(\frac{x'}{a} + \alpha\right)\right\} + (b^2 - a^2)(x - x') = 0,$$

and that if x' corresponds to the nth intersection from x, $x - x'$ is $> (n-1)\pi a$ but $< n\pi a$.

26. Prove that in a transparent medium whose refractive index μ is a function of x the distance from a plane, the curvature of a ray at a point of its course where it is parallel to that plane is $= \frac{1}{\mu}\frac{d\mu}{dx}$.

27. If the path of a ray of light be $y = a\epsilon^{\frac{x}{c}}$ shew that the index of refraction at any point is determined from the equation

$$\mu = (y^2 + c^2)^{\frac{1}{2}} f\left(x + \frac{y^2}{2c}\right).$$

28. A point of light is placed at the origin of co-ordinates in a medium where the refractive index is given by $\mu = e^{-\kappa x}$. Shew that an eye placed at the point (x, y) will see the origin of light by means of a small pencil of light—one of the focal lines of which lies on the axis of x and the other at a distance

v from the eye, where $\kappa^2 v^2 e^{\kappa x} \cos^2 \kappa y = 2 (\cos \kappa x - \cos \kappa y)$—whilst the axis of the pencil makes an $\angle \psi$ with the axis of x, where $\cot \psi \sin \kappa y = \cos \kappa y - e^{-\kappa x}$.

29. A ray is propagated through a medium of variable density in a plane with respect to which the medium is symmetrical, prove that the differential equation of the path is

$$\mu \frac{dp}{dr} = p\left(\frac{d\mu}{dr}\right) - \frac{\sqrt{r^2 - p^2}}{r} \left(\frac{d\mu}{d\theta}\right).$$

Find the general form of μ if this path be the equiangular spiral $p = r \sin \alpha$;—and prove that along the spiral $\mu \propto r$.

30. The refractive index at any point of a spherical shell is a function $f(r)$ of the distance r from the centre,—and a pencil of light, proceeding from a point at a distance c from the centre, is directly refracted through the shell; shew that the distance (c') from the centre of the geometrical focus after emergence is given by the equation

$$\tfrac{1}{2}\{f(a)\}^2\left(\frac{1}{c'} - \frac{1}{c}\right) = \frac{f(b)\{f(b) - 1\}}{b} - \frac{f(a)\{f(a) - 1\}}{a} - \int_b^a \frac{f'(r)}{r} \, . dr,$$

where a, b, are respectively the external and internal radii of the shell, and c' is measured in the same direction as c.

Reflexion, &c. in any manner.

31. A ray of light whose direction-cosines referred to three rectangular axes are λ, μ, ν is incident at an angle ϕ on a reflecting plane

$$lx + my + nz = 0;$$

shew that the direction-cosines of the reflected ray are equal to

$$2l \cos \phi - \lambda, \quad 2m \cos \phi - \mu, \quad 2n \cos \phi - \nu.$$

32. A ray is reflected at two plane mirrors so that the planes of reflexion are at right angles; shew that the deviation is a minimum when the angles of incidence are equal.

And if ϕ be the angle of incidence, D the minimum deviation, i the inclination of the mirrors, shew that

$$\cos i = \cos^2 \phi, \quad \text{and } \cos D = \cos^2 2\phi.$$

33. Prove that when a ray of light is reflected at any curve-surface the length of the course of the ray, reckoned from any point in the incident ray to any point in the reflected ray, is a minimum.—

Camb. and Dub. Math. Journal, Vol. II. p. 286.

If a polished plane have an indefinite number of very fine concentric circular grooves turned on its surface, and light be incident on it from a luminous point, the appearance presented to the eye of an observer will be that of a bright curve; find its equation.

34. If a ray of light be reflected once at each of two plane surfaces in any planes, shew that

$$\cos D = \cos 2I + 2 (\sin I \sin \phi \sin \theta)^2,$$

where D is the whole deviation of the ray, I the angle between normals to the surfaces, ϕ the angle of incidence of the first incident ray, and θ the angle between the plane of incidence of the first incident ray and a plane perpendicular to the intersection of the reflecting surfaces.

35. There are three plane reflectors, two of which are at right angles to each other, and a ray of light is incident upon the third and reflected successively by each of them;—it is required to shew that the angle between the first incident and last reflected rays is equal to twice the angle of incidence upon the first surface.

36. An oblique cone of rays falls upon a narrow reflecting annulus cut from the surface of a cone by planes perpendicular to the axis; find the general form of the reflected image thrown on screens at different distances perpendicular to the axis of the cone.

37. A ray of light is incident parallel to the axis of a reflecting elliptic paraboloid whose equation is

$$\frac{y^2}{b}+\frac{z^2}{c}=2x$$

at a point (α, β, γ); shew that the equations of the reflected ray are

$$\frac{x-\alpha}{\left(\frac{\beta}{b}\right)^2+\left(\frac{\gamma}{c}\right)^2-1}=\frac{y-\beta}{\frac{2\beta}{b}}=\frac{z-\gamma}{2\gamma}.$$

Each reflected ray will pass through each of two parabolas lying in the principal planes of the paraboloid.

38. A ray is incident at a point x, y, z of a surface $u=0$, and there is both a reflected (i) and refracted (ii) ray,—if l, m, n be the direction-cosines of the normal to the surface $u=0$ at x, y, z,—$\alpha\beta\gamma$ those of the incident ray, i the angle of incidence and reflexion, i' that of refraction—the equations to the reflected and refracted rays are severally

$$\frac{X-x}{2l\cos i-\alpha}=\frac{Y-y}{2m\cos i-\beta}=\frac{Z-z}{2n\cos i-\gamma}\text{...........(i)},$$

$$\frac{X-x}{l\sin(i-i')+\alpha\sin i'}=\frac{Y-y}{m\sin(i-i')+\beta\sin i'}=\frac{Z-z}{n\sin(i-i')+\gamma\sin i'}\text{ (ii)}.$$

39. A ray of light is incident from the centre of an ellipsoid, the inner surface of which is polished, and the equation of which is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0;$$

prove that the equations to the reflected ray will be

$$\frac{X-x}{x\left(2\frac{p^2}{a^2}-1\right)}=\frac{Y-y}{y\left(2\frac{p^2}{b^2}-1\right)}=\frac{Z-z}{z\left(2\frac{p^2}{c^2}-1\right)},$$

where x, y, z are the co-ordinates of the point of incidence, and

$$\frac{1}{p^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}.$$

Further,—Prove that all rays which after reflexion pass through the line $x = y = z$, were before reflexion in the surface of the cone defined by the equation

$$yz\left(\frac{1}{b^2} - \frac{1}{c^2}\right) + zx\left(\frac{1}{c^2} - \frac{1}{a^2}\right) + xy\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = 0.$$

40. If a ray be reflected at two plane surfaces, its direction before incidence being parallel to the plane bisecting the angle between the two planes, and making an angle θ with their line of intersection, prove that $\sin\frac{D}{2} = \sin\theta\sin\alpha$, α being the angle between the planes and D the deviation.

41. A pencil of rays is incident parallel to the axis x of a refracting ellipsoid whose greatest and least axes are a and c respectively; find the nature and *limits* of the two curvilinear boundaries of the portion of the plane x, y through which all the rays will pass;—and shew that if e, e', e'' be the eccentricities of the principal sections in the planes xy, xz, yz respectively, and $\mu = \frac{1}{e}$, the boundaries will be two ellipses whose semi-axes are ae''^2, be''^2, and $\frac{cae}{b}$, be''; but if $\mu = \frac{1}{e'}$, all the rays will pass through a *portion* of the *arc* of an ellipse, whose semi-axes are ae', be'' included between the vertex and a double ordinate at a distance $\frac{cae'}{b}$ from its centre.

42. The surface of a piece of water is covered, except one narrow slit in the form of a straight line; a luminous point is placed in a given position above the surface;—if this be taken as origin of co-ordinates and the vertical line as axis of z; and $x = a$, $z = -c$ are the equations to the slit,—shew that the equation to the sheet of light in the water is

$$a^2\{(x^2 + y^2)(x - a)^2 + x^2(z + c)^2\} = \mu^2(x - a)^2\{a^2(x^2 + y^2) + c^2x^2\}.$$

43. Rays diverging from a point a, b, c are reflected at the curve $z=0, f(x, y)=0$, the normals to the polished part of which all lie in the plane $z=0$; shew that the equations to the ray reflected at the point $(x, y, 0)$ are

$$\frac{X-x}{2p(b-y)-(p^2-1)(a-x)}=\frac{Y-y}{2p(a-x)+(p^2-1)(b-y)}=\frac{Z}{(p^2+1)c}$$

where $p=\frac{dy}{dx}$,—and the sheet or surface formed by the reflected rays will be found by eliminating x, y between these equations and $f(xy)=0$.

Ex. If $f(x, y)=y^2-4ax$ and $b=0$, the ray surface of reflexion is

$$Y^2(c-Z)=4a(aZ+cX).$$

* * * * *

The following examples of *ray-surfaces of reflexion* formed as in problem (43) are taken with modifications from an interesting work on *Ray Surfaces of Reflexion*, by Prof. Childe, Cape Town, 1857. I am indebted to the same work for the problems 51—54 of this Chapter. *Lateral* and *vertical* ray-surfaces of reflexion are the names given to surfaces formed as in problems 43 and 51 respectively.

44. In problem (43)

(i) The equation to the plane passing through the incident ray and the normal to the reflecting curve is

$$X-x+(Y-y)p-\{a-x+(b-y)p\}\frac{Z}{c}=0.$$

(ii) The equation to the plane passing through the reflected ray and the tangent to the reflecting curve is

$$(X-x)p-(Y-y)+\{(a-x)p-(b-y)\}\frac{Z}{c}=0.$$

45. In (43) if the incident rays are all parallel, their direction being defined by the direction-cosines l, m, n, the equations of the reflected ray become

$$\frac{X-x}{2mp-l(p^2-1)}=\frac{Y-y}{2lp+m(p^2-1)}=\frac{Z}{(p^2+1)n}.$$

In this example, if the ratio $l : m$ remain constant whilst n varies, the resulting ray-surfaces of reflexion will have for their envelope the cylinder whose axis is parallel to the axis of z, and whose trace on the plane of xy is the ordinary caustic formed by the reflexion of rays all incident in the direction defined by (l, m, o).

•46. If $f(xy) = y - x \tan \alpha = 0$, the ray-surface of reflexion (either *lateral* or *vertical*) is

$$\frac{Z}{c} + \frac{X \sin \alpha - Y \cos \alpha}{a \sin \alpha - b \cos \alpha} = 0.$$

47. If $f(xy) = x^2 + y^2 - \rho^2 = 0$ a circle,—the ray-surface of reflexion is

$$\{c^2(x^2+y^2) - (a^2+b^2)z^2\}^2 = \rho^2 (z+c)^2 \{(cx-az)^2 + (cy-bz)^2\}.$$

48. In the preceding example if the incident rays are all parallel, and their directions defined by (l, m, n), the ray-surface of reflexion is

$$\text{(i)} \quad \{n^2(x^2+y^2) + (l^2+m^2)z^2\}^2 = \rho^2 \{(nx-lz)^2 + (ny-mz)^2\},$$

or writing $m = 0$—which does not sacrifice any generality of form—this becomes

$$\{n^2(x^2+y^2) - l^2z^2\}^2 = \rho^2\{(nx-lz)^2 + n^2y^2\}\ldots\ldots\ldots$$

(ii) If the direction of this system of parallel rays be varied, subject to the conditions $m = 0$, $l^2 + n^2 = 1$, only—the envelope of the ray-surfaces given by (i) is

$$(\rho^2 - x^2 - y^2)\{8(x^2+y^2) + \rho^2\}^2 = 27\rho^4 x^2.$$

This surface is an epicycloidal cylinder whose axis is parallel to the axis of z, and whose trace on xy is the epicycloid generated by a circle of radius $\frac{\rho}{4}$ rolling on a circle of radius $\frac{\rho}{2}$; the cusps of the epicycloid being on the axis of x. It is in fact the epicycloid of Art. 71.

49. Find the envelope of the *lateral* ray-surfaces of reflexion, the reflecting curve being (as in 47)—the circle

$$x^2 + y^2 - \rho^2 = 0,$$

and the locus of the radiant point (a, b, c) a circular ring defined by $c = \text{constant}$, $a^2 + b^2 = \kappa^2$.

Result. The double cone given by the equation

$$c\sqrt{(x^2+y^2)}+\kappa z=\pm\rho\,(z+c).$$

50. If the reflecting curve (as in 43) be a cycloid

$$z=0, \qquad y=a\,\text{vers}^{-1}\frac{x}{a}\sqrt{(2ax-x^2)}\,;$$

and the incident rays are all parallel to the plane of xz, their directions being defined by (l, O, n)—the *lateral* ray-surface of reflexion is

$$ny=na\,\text{vers}^{-1}\left(\frac{lz+nx}{lz+na}\right)+\sqrt{[(lz+nx)\,\{lz-n\,(x-2a)\}]}.$$

This surface is discussed at length in the work above referred to.

* * * * *

51. Rays diverging from a point (a, b, c) are reflected at a polished curve whose equations are $z=0, f\,(x,y)=0$, the normals to which are all parallel to the axis of z: shew that the equation to the surface formed by the reflected rays—*the ray-surface of reflexion*—is

$$f\left(\frac{cX+aZ}{c+Z},\ \frac{cY+bZ}{c+Z}\right)=0.$$

52. In the preceding question, if the incident rays are all parallel—their direction being defined by the direction-cosines l, m, n—the ray-surface of reflexion will be

$$f\left(\frac{nX+lZ}{n},\ \frac{nY+mZ}{n}\right)=0.$$

53. If the polished curve in problem (51) be a circle, *viz.*

$$f\,(xy)=0=x^2+y^2-\rho^2,$$

the ray-surface of reflexion is

$$(cx+az)^2+(cy+bz)^2=\rho^2\,(z+c)^2.$$

54. Find the envelope of the ray-surfaces of reflexion

formed by the curve $z = 0$, $x^2 + y^2 - \rho^2 = 0$, as in problems 51 and 53, when the locus of the radiant point a, b, c is

(i) A circular ring defined by the equations $c =$ constant, $a^2 + b^2 = k^2 =$ a constant.

Result. The double cone defined by the equation

$$c\sqrt{(x^2 + y^2)} + xz = \pm \rho\ (z + c).$$

(ii) A circular ring defined by $c =$ const., and

$$(a - \alpha)^2 + (b - \beta)^2 = \kappa^2,\ \alpha,\ \beta,\ \kappa \text{ constants.}$$

Result. Two oblique cones defined by

$$(cx + \alpha z)^2 + (cy + \beta z)^2 = \{(\rho \pm \kappa)\ z + \rho c\}^2.$$

(iii) A luminous spherical surface concentric with the reflecting circle and defined by the equation $a^2 + b^2 + c^2 = \kappa^2$.

Result. Two right cones defined by the equation

$$\{\sqrt{(x^2 + y^2)} \pm \rho\}^2 = \frac{\rho^2 - \kappa^2}{\kappa} \cdot z^2.$$

(iv) A luminous paraboloid of revolution, *viz.*

$$a^2 + b^2 = 4mc.$$

Result. The envelope is a cone,

$$\rho^3\ (x^2 + y^2) = (mz - \rho^2)^2.$$

CHAPTER VII.

Spherical aberration of lenses;—Excentrical pencils.

1. A pencil of parallel rays is directly refracted through an equiconvex lens of 20 inches focal length and 4 inches aperture, ($\mu = 1{\cdot}5$) — the aberration of the extreme rays $= \frac{1}{3}$ inch.

2. A man stands opposite to a convex mirror fixed to one of the walls of a room; what appearance as to size and position will his image, seen in the mirror, present to him?

3. If the thickness of a concavo-convex lens be equal to $(\mu+1)$ times the distance between the centres of its spherical surfaces; shew that a point may be found in its axis, from which if rays diverge and fall upon the concave surface, they will diverge accurately from a point after emergence.

4. The form of a lens $\left(\mu=\frac{3}{2}\right)$ being such that the aberration for parallel incident rays being a minimum,—express the aberration in terms of the focal length of the lens and the breadth of the incident pencil.

5. A pencil is refracted directly through an equiconvex lens, of focal length f, the origin and geometrical focus being equidistant from the centre,—the magnitude of the aberration

$$=\frac{9y^2}{2f}\ldots\left(\mu=\frac{3}{2}\right).$$

6. When a small pencil of parallel rays is centrically and obliquely refracted through a thin lens, the magnitude of the circle of least confusion is unaffected by the form and substance of the lens, and depends only on the breadth of the incident pencil, and the inclination of its axis to that of the lens.

7. A direct pencil of rays converges to a point at the extreme distance from a lens at which an absence of aberration in a converging pencil is possible: determine the relation between the radii of the lens that there may be no aberration in the refracted pencil. $\left(\mu=\frac{3}{2}\right)$.

Result. With the notation of Art. 132, we must have

$$\frac{\mu+2}{\mu-1}x^2-4(\mu+1)ax+(\mu-1)(3\mu+2)a^2+\frac{\mu^3}{\mu-1}=0,$$

which gives

$$x=\frac{2(\mu^2-1)a\pm\mu\sqrt{(\mu-1)^2a^2-\mu(\mu+2)}}{\mu+2}.$$

the limiting value of a is $\quad a=\pm\frac{\sqrt{\mu(\mu+2)}}{\mu-1}=\pm\sqrt{21}=\pm 4{\cdot}58$

the pencil being convergent at incidence, u is negative, and therefore the negative value of a must be taken, and we obtain

$$x = 2\frac{\mu^2 - 1}{\mu + 2}a = -3{\cdot}27,$$

$$\text{and } \frac{r}{s} = \frac{x-1}{x+1} = \frac{427}{227} = \frac{32}{17} \text{ nearly.}$$

8. A pencil of parallel rays falls directly upon an equiconvex lens: shew that if the lens be split in two by a plane perpendicular to its axis and the two parts be separated by an interval equal to $\frac{2}{3}$ the focal length of either, the convergency of the emergent pencil is double that in the former case when the lens is entire.

9. Three convex lenses are placed on the same axis; shew that the focal length (F) of the equivalent single lens is given by the equation

$$f_1f_2 + f_2f_3 + f_3f_1 - a(f_2 + f_3) - b(f_1 + f_2) + ab = \frac{f_1.f_2.f_3}{F},$$

where $f_1.f_2.f_3$ are the focal lengths of the three lenses, and a, b the distances between the first and second, and the second and third respectively.

10. Two lenses of the same material ($\mu = 1{\cdot}5$) the radii of whose faces taken in order are $r_1s_1r_2s_2$ are placed on the same axis at a distance a. The conditions that the system may be achromatic for a direct pencil of parallel rays and at the same time that the spherical aberration at each lens may be a minimum are given by the equations

$$s_1 = -6r_1, \quad (70a - 24r_1)\,r_2 = (70a + 144r_1)\,s_2 = (7a + 12r_1)^2.$$

11. A pencil of parallel rays whose axis before refraction cuts the common axis of two lenses (focal lengths f_1, f_2 at a distance a) at a distance d from the centre of the first lens—the focal length F of the equivalent simple lens is given by the equation

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} + \frac{a}{f_2}\left(\frac{1}{f_1} + \frac{1}{d}\right.$$

12. A plano-convex lens is in contact with a concavo-plane lens on the same axis, the refractive indices being μ, μ', and r the radius of the common spherical surface. A ray which cuts the axis at a small $\angle\ \epsilon$, and at a distance d from the compound lens, is refracted through it. Prove that the deviation of the ray is $(\mu - \mu')\dfrac{d}{r}\epsilon$.

13. Two lenses, the thicknesses of which are t, t', radii r, s, r', s', and indices μ, μ', having a common axis, and touching each other in the axis, are traversed by a ray so that its directions before incidence and after emergence are parallel.

Shew that if ω be the distance of the ultimate intersection of the axis and the path of the ray between the two lenses, from the point of contact of the lenses,

$$\left\{\frac{\mu-1}{\mu}\cdot\frac{t}{r}+\frac{\mu'-1}{\mu'}\cdot\frac{t'}{s'}\right\}\frac{1}{\omega}$$

$$=(\mu-1)\left(\frac{1}{r}-\frac{1}{s}-\frac{\mu-1}{\mu}\cdot\frac{t}{rs}\right)+(\mu'-1)\left(\frac{1}{r'}-\frac{1}{s'}-\frac{\mu'-1}{\mu'}\cdot\frac{t'}{r's'}\right).$$

CHAPTER VIII.

Images.

1. A lighted candle is placed before a concave spherical mirror, the candle being perpendicular to the axis of the mirror and in the same plane with it; find how the image of the flame moves as the candle burns.

2. A short object is placed perpendicularly on the axis of a concave spherical refractor, and at a distance from it equal to $\dfrac{f}{\mu}$, f being the focal length; prove that the linear magnitude of the virtual image is half that of the object.

3. A person regards the image of himself in a large concave mirror, trace the changes of the image in magnitude,

position, and distinctness, as he walks from a considerable distance along the axis up to the surface.

4. The distance of a luminous point from a concave spherical reflector is less than half its radius; determine (i) the points of incidence at which consecutive reflected rays will be parallel, (ii) the position of the luminous point along the axis which renders the distance of these points from the axis a *maximum* or *minimum*.

5. A circular luminous ring, radius a, is placed on the ceiling of a room;—in the floor is a hole in the form of an equilateral triangle of side b; shew that the area of illuminated patch on the floor of the room beneath

$$= \pi a^2 + \sqrt{3}b^2 + 6ab,$$

the rooms being of equal height.

6. PSp is a focal chord of a parabola, S a luminous point; shew that the illumination at P : illumination at $p :: (Sp)^{\frac{5}{2}} : (SP)^{\frac{5}{2}}$.

7. Two candles (P, Q) of equal intensity situated at the same distance from a wall, throw upon it shadows (m, n) of a small object; prove that the illumination in these two shadows are as $Pn^3 : Qm^3$.

8. Two candles are placed on a horizontal table, the one lighted, the other not,—the latter being the taller and fixed: the other is moved about on the table so that the shadow of the first on the ceiling is of constant length. Shew that the locus of the foot of the candle is a conchoid.

9. The circumference of a circle is luminous except a very small arc which serves as a reflector; find the image of the luminous part when the image is defined to be the locus (i) of the primary focal lines, (ii) of the secondary focal lines, (iii) of the circles of least confusion.

10. The curvature at the vertex of the image of a straight

line formed by refraction through a sphere $= 2\dfrac{\mu - 1}{\mu}$. curvature of the sphere.

11. A parabola whose latus rectum $= 2l$ is placed before a concave spherical refracting surface, the focus and axis of the parabola being coincident with the centre and axis of the surface. The parabola is convex to the surface, and an image of it is formed by direct pencils. Shew that if f be the distance of the principal focus from the surface (f being $> l$) the image will be a confocal hyperbola whose eccentricity is $= \dfrac{f}{f-l}$.

12. In the flat roof of a building there is a square hole exposed to the light of a uniform sky. The floor of the uppermost room has also a square hole in it under the former but its angles opposite to the sides of the former:—determine the form of the whole illuminated surface of the floor of the room below, and the illumination at any point.

13. A straight line passes through the point of intersection of three rectangular reflecting planes, and is equally inclined to each of them; find the series of images produced, and distinguish between those resulting from one, two, and three reflexions respectively.

14. The image of a straight line perpendicular to the axis of a convex lens at a very great distance from it, approximates to a parabolic curve, whose equation is

$$2(x+f) + \left(2 + \frac{1}{\mu}\right)\frac{y^2}{f} = 0,$$

the centre of the lens being the origin, and the circle of least confusion being taken to be the image of any point.

15. Shew that the image of the surface $f(x, y, z) = 0$ made by reflexion at the plane mirror $lx + my + nz = 0$ is $f(x + lp, y + mp, z + np) = 0$, p being such a quantity that $lx + my + nz + \dfrac{p}{2} = 0$.

16. An image of a very distant object is formed by a plano-convex lens, the pencils before incidence on the lens passing through a small diaphragm (B), whose centre is in the axis (ABO) of the lens: O being the centre of the convex surface, and A the point where the axis meets the plane surface of the lens. If $AO = \mu . AB$, the image formed by the lens will be *distinct*.

N.B. *Distinctness* will be secured if the refraction at the second surface—the *convex* one—be *direct*.

17. A stop is placed on the axis of a concave spherical reflector at a distance from it equal to four times its focal length: to measure the distortion at any point of the image of a very distant object to which the axis is directed.

With a figure and notation similar to that of Art. 150 (the position of E and Y must be reversed)—if Y be the stop or diaphragm, $AR = y$, q being the point of the image seen corresponding to the point Q of the object, we shall obtain

$$qm = \frac{r}{2} \tan \epsilon \left(1 + \frac{3y^2}{8r^2}\right),$$

now the part of the object whereof qm is the image has a size proportional to $\tan \epsilon$; hence the distortion at any point of the image is measured by the term involving y^2 in the expression

$$\frac{r}{2}\left(1 + \frac{3}{8}\frac{y^2}{r^2}\right).$$

18. A small light is placed at the focus of a perfect reflector in the form of a paraboloid of revolution: prove that the brightness, due to reflexion, at any point within the volume of the paraboloid, varies inversely as the square of the focal distance of the end of the diameter through the point.

19. The caustic by reflexion at the parabola $2y^2 = 3cx$, the vertex being the luminous point is represented by the equation $8c^3x - 6c^2y^2 - y^4 = 0$.

20. If S be the luminous point, SZ the polar subtangent of the curve (Art. 153), ZN the perpendicular on the corresponding tangent to the caustic, then the locus of N is an involute of the caustic.

21. A glass scale divided into equal parts is placed at a distance a from a concave mirror at right angles to the axis;

to an eye placed at a distance b beyond the scale, n divisions of the scale appear to cover $n+1$ divisions of its inverted image; shew that if r is the radius of the mirror

$$\frac{1}{r}=\frac{1}{a}+\frac{b}{2na(a+b)}.$$

22. In Art. 153. If $Pp=\frac{1}{n}.SP$ for all points of the curve of reflexion, shew that the curve of reflexion is

$$\left(\frac{r}{c}\right)^{\frac{n-1}{2}}=\sin\frac{n-1}{2}\theta.$$

23. If the origin of light be on the surface of the sphere, the caustic surface formed by rays which have undergone n reflexions is the surface of revolution whose generating curve is an epicycloid formed by a circle of radius $\frac{na}{2n+1}$ rolling on a circle of radius $\frac{a}{2n+1}$.

24. If the origin be the source of light, and the distance between the point of incidence and the corresponding point in the caustic by reflexion be $\frac{\phi(r)}{\phi'(r)}$, shew that the form of the reflecting curve is given by

$$\left(\frac{dr}{d\theta}\right)^2+r^2=\frac{cr^3}{\phi(r)}.$$

25. The focal length of a double equiconcave lens ($\mu=\frac{3}{2}$) is 5 inches; prove that the distance from the lens of images of a distant object formed (i) by reflexion at the 1st surface, (ii) by one reflexion at the 2nd surface, (iii) by two reflexions at the 2nd surface are $2\frac{1}{2}$ inches, $1\frac{1}{4}$ inches and $\frac{1}{2}$ inch respectively:—and compare the sizes of the three images.

26. At a point on the inside of a polished hollow circular cylinder of radius a is placed a luminous point: explain the formation of a series of bright curves on any plane at right angles to the axis of the cylinder; and prove

that they are all epicycloids, the radius of the rolling circle for the nth curve being $\frac{na}{2n+1}$, and that of the fixed circle $\frac{a}{2n+1}$.

27. A luminous point moves along a diameter of a reflecting circle of radius a; prove that the two cusps of the caustic, which are not on that diameter, move on the curve

$$r = a \cos \frac{\theta}{2}.$$

28. Parallel rays fall on a curve whose equation is given in the form $s = f(\psi)$, where ψ is the angle the normal makes with a fixed line parallel to the direction of the rays. Shew that the radius of curvature of the caustic is given by the equation

$$4\rho' = f''(\psi) . \cos \psi + f'(\psi) \sin \psi.$$

29. A bright point moves along the directrix of a parabola $y^2 = 4ax$, capable of reflecting it; prove that the locus of its image is a curve whose equation is

$$27ay^2 = (2x - a)(5a - x)^2.$$

30. Light is transmitted from a very distant luminous sphere through an orifice in an opaque screen A, and is received on a second screen B parallel to A:—shew that

(i) if the distance of B from A be small, compared with the size of the orifice, the illuminated figure on B will resemble the orifice in A.

(ii) if the distance of B from A be large, the illuminated figure on B will be nearly circular.

Caustics.

31. The perimeter of an ellipse is luminous except a very small arc at the extremity of the major axis, determine the caustic.

32. If light be incident from a point Q on any surface, and if x be the half chord of curvature at any point of the surface in the direction of the incident ray, then $\frac{x}{2}$ is a mean proportional between $u - \frac{x}{2}$ and $v - \frac{x}{2}$.

If the equation to the surface be given in the form $u = \phi(x)$, then the arc of the caustic can be found from the equation

$$(2u - x)^2 \, ds = 4u \, (u - x) \, du + 2u^2 dx.$$

33. Parallel rays are incident parallel to an asymptote on a reflecting rectangular hyperbola; find the equation to the caustic curve.

If ρ be the radius of curvature of the caustic, ρ_1 of the corresponding point of the hyperbola, shew that the brightness of the caustic $\propto \dfrac{\rho}{(\rho_1)^{\frac{4}{3}}}$.

34. Rays emanate from the pole of a plane curve whose equation is given in the form $f(r, p) = 0$ (i); shew that the equation to the catacaustic will be the result of eliminating r and p between (i) and the equations

$$\sqrt{(r'^2 - p'^2)} + \sqrt{(r^2 - p'^2)} = \frac{rp\dfrac{dr}{dp}}{2r - p\dfrac{dr}{dp}} \text{..........(ii)},$$

$$p'^2 = \frac{4p^2(r^2 - p^2)}{r^2} \text{......................(iii)}.$$

Conversely, if $\phi(r', p') = 0$, (iv), be the equation to the catacaustic, the equation of the reflecting curve will be obtained by eliminating r' p' between (ii), (iii), and (iv).

Examples.

(i) If $r^2 = p^2 + a^2$,—the involute of a circle,—be the reflecting curve, the caustic is $\dfrac{r}{2a} = \dfrac{\sqrt{\{16a^4 + p^2(4a^2 - p^2)\}}}{8a^2 - p^2}$.

(ii) If $r\theta = a$, or $(r^2 + a^2)\, p^2 = a^2 r^2$,—the hyperbolic spiral, —be the reflecting curve, the caustic is $r = \dfrac{ap}{a - p}$.

(iii) If $r = a$—a circle—be the equation to the caustic, the equation to the reflecting curve is $ar = 2p\sqrt{(r^2 - p^2)}$.

35. If rays emanating from the vertex are reflected from a parabola, the caustic is the evolute of a cissoid.

36. Rays emanating from the focus of a parabola are reflected from the evolute of the parabola—the caustic is the evolute of a parabola.

37. The caustic of a cardioid $r = a(1 + \cos\theta)$,—the pole being the radiant point,—is an epicycloid due to a circle of radius $\frac{a}{4}$ rolling on a circle of radius $\frac{a}{2}$.

Note. A neat geometrical proof of this problem is given by Dr A. H. Curtis in the *Messenger of Mathematics*, new series, No. 135, July 1882.

The same problem in substance was proposed by Prof. J. C. Adams in a Smith's Prize paper, 1860, in the following form:

A surface is generated by the revolution of a cardioid round the chord with respect to which it is symmetrical; if a point of light be placed in the cusp of the surface, shew that the equation to the caustic curve is

$$27a^4r^2\sin^2\theta = 4(r^2 - a^2)^3,$$

the distance of the pole from the cusp being a, which is one-fourth of the length of the axis, and the initial line coinciding with the axis.

38. A ray whose direction is parallel to the axis of x is incident on a point xy of a refracting curve; prove that the corresponding point of the caustic $(x'y')$ is given by the equations

$$x' = x - \frac{\mu^2(1+p^2) - 1}{\mu^2 q}\left[\frac{1 + p\sqrt{\{\mu^2(1+p^2) - 1\}}}{\sqrt{\{\mu^2(1+p^2) - 1\}} - p}\right],$$

$$y' = y + \frac{\mu^2(1+p^2) - 1}{\mu^2 q},$$

where $p = \frac{dy}{dx}$, $q = \frac{d^2y}{dx^2}$.

A ray whose direction is parallel to the axis of x is incident on a parabola $x^2 = 2ay$ at the point $x = y = 2a$, and is refracted into a medium for which $\mu = \sqrt{2}$,—find the corresponding point on the caustic.

CHAPTER IX.

Of the Chromatic Dispersion of Light.

1. In Newton's experiment—supposing a second prism to be placed behind the first with its edge perpendicular to it, what will be the effect upon the spectrum?

2. When objects are viewed by means of pencils of light which are refracted through an isosceles prism, suffering an internal reflexion at the base, the objects are seen quite free from chromatic dispersion, provided the prism be not very large; prove that this must happen in the general case,—that is, when the refraction is not supposed to take place in a plane perpendicular to the edges of the prism.

3. If a prism be laid on its base in the open air and the eye be placed in a proper position, the base will appear to be divided into two parts, the one much brighter than the other, and separated from one another by a coloured bow (*blue and violet*) concave towards the eye. Give *Newton's* explanation of the cause of these phenomena.

Newton, *Optics*, Book I. Part II. Prop. 8.

4. An opaque rod held parallel to the edge of a glass prism is viewed by an eye applied to the prism,—the rod appears curved with its concavity towards the edge,—and fringed with red and yellow on its concave side, and with violet and blue on the convex side. Explain these phenomena.

5. Two plates of different dispersive powers, but in both of which the index of the extreme violet rays is the same, are placed in contact, and light is obliquely incident upon them. Prove that the reflected rays will be tinged with red, and the transmitted rays with violet.

6. A black line is painted on the base of an isosceles prism, midway between its two edges, and the prism is then interposed between the sun-light and an eye held at the edge of the prism looking towards the base. Describe the appearances presented with respect to position and colour.

7. A luminous point of white light being placed on the axis of a lens,—shew that as the point moves along the axis the distance between the geometrical foci for two given colours will pass through a minimum,—and determine the position of the point in this case.

8. A star, the altitude of which is α, is viewed by an eye under water; shew that it will appear in the form of a spectrum, with the violet end uppermost, and that the angular magnitude of the spectrum will be

$$\sin^{-1}\left(\frac{\cos\alpha}{\mu_r}\right) - \sin^{-1}\left(\frac{\cos\alpha}{\mu_v}\right)$$

9. An achromatic object-glass, whose focal length is 70 inches, is composed of two lenses of crown and flint-glass respectively; find the focal length of each lens, the dispersive powers of crown and flint-glass being in the ratio of 7 : 10.

10. For a prism A of small refracting angle the indices for two colours are 1·2 and 1·22, and for a prism B are 1·5 and 1·57; find the angles of the prisms in order that the common deviation of the two colours—passing through the prisms in a principal plane of each—may be $= 1^\circ$.

11. A system of three convex lenses of the same substance on the same axis will be achromatic for rays incident parallel to the axis, if

$$3ab - 2a(f_2 + f_3) - 2b(f_1 + f_2) + f_1 f_2 + f_2 f_3 + f_3 f_1 = 0;$$

a, b being the distances of the middle lens—focal length f_2—from the two extreme ones whose focal lengths are f_1, f_3.

12. A ray of light after passing through a prism at the angle of least deviation falls perpendicularly on the surface of another prism of different substance, shew that the ray will be achromatised if the angle between the prisms

$$= \cos^{-1}\left(2m \sin\frac{i}{2} \cdot \cot i_1\right),$$

where m is the ratio of the indices of refraction and i, i_1 the refracting angles of the prisms—the ray passing in a principal plane of each.

13. A small pencil of rays of common light is incident obliquely on a plane refracting surface, shew that the primary foci of the refracted pencils of different colours will lie on a curve of the third order.

14. Find the condition of achromatism of an eye-piece composed of a solid block of glass with a thin lens of different glass cemented to it on the end next the eye,—the surfaces being worked spherical.

15. Find the condition of achromatism of an eye-piece composed of a single block of glass worked at the ends into spherical surfaces and used with an object-glass of great focal length.

16. A compound object-glass is to be formed of two lenses in contact; shew that if when the lenses are ground, achromatism is nearly but not quite secured, the defect may be remedied by slightly separating the lenses.

The refractive indices corresponding to the letters D and F in the *orange* and *blue*, for certain kinds of crown and flint glass are

Crown-glass . . .	1·5279,	1·5344,
Flint-glass . . .	1·6351,	1·6481,

twenty inches is to be the focal length of the proposed object-glass; find the focal lengths of the two lenses which, placed in contact, unite these *lines*.

17. Trace the variation in the angular magnitude of the spectrum formed by a prism, for different angles of incidence, and shew that it is a minimum when, with the usual notation,

$$\mu^2 \sin(\phi' - i)\cos(2\phi' - i) + \sin\phi' = 0.$$

18. If a ray passes through two prisms whose edges are parallel, with minimum deviation through each, and be

achromatic at emergence, then will

$$\frac{d\mu}{\mu} . \tan \phi + \frac{d\mu_1}{\mu_1} \tan \phi_1 = 0,$$

where ϕ, ϕ_1 are the angles of incidence and emergence.

19. Why are we entitled to assume

$$\frac{\Delta\mu}{\mu - 1} = a \frac{\Delta x}{x - 1} + b \left(\frac{\Delta x}{x - 1}\right)^2 + c \left(\frac{\Delta x}{x - 1}\right)^3 + \ldots$$

$\frac{\Delta x}{x - 1}$, $\frac{\Delta\mu}{\mu - 1}$ being the dispersive powers of *water* and *any other medium?* Determine by means of this equation the ratio of the refracting angles of an achromatic combination of n prisms of small refracting angles all nearly in their position of minimum deviation.

Herschel's *Light*, Art. 439...

20. A luminous point is placed on the axis of a double convex lens at a greater distance than its focal length. What will be the appearance on a screen placed—say—in the geometrical focus of the *green* rays? Taking 2 and -2 inches for the radii of the lens, 3 inches for the distance of the luminous point, $\mu_v = 1{\cdot}344$, $\mu_r = 1{\cdot}331$,—find the dimensions of the least circle of chromatic aberration.

21. A cylinder is composed of a transparent medium for which the critical angle which is $> \frac{\pi}{4}$ lies between $\tan^{-1}\frac{r}{a}$ and $\tan^{-1}\frac{a}{r}$. Supposing there to be no internal reflexion, shew that to an eye placed close to the centre of a circular end there will appear a dark ring fringed with colours;—indicate the colours which will compose the fringes,—a the altitude of the cylinder being greater than r the radius of the base.

22. When a spectrum is measured in Fraunhofer's manner; to find the angle subtended through the telescope be-

tween the *fixed line A* in its axis, and the *fixed line B*, having given $\mu_A = \sqrt{3}$, $\mu_B - \mu_A = \cdot 001$, the refracting angle of the prism being $= 60^\circ$ and the distance of its edge from the slit and the object-glass being 3 and 10 times the focal length of the object-glass respectively,—the telescope magnifying 20 times.

With the notation of Art. 168, the angle which the lines A and B subtend at the prism $= \dfrac{\sin i}{\cos \phi' \cos \psi} \delta\mu = \cdot 002$ in circular measure, after reduction.

If $f =$ focal length of the object-glass, the linear distance of A and B in the image formed by the prism $= 3f.(\cdot 002) = \cdot 006 . f$. This distance subtends at the object-glass, the $\angle \dfrac{\cdot 006}{13}$, and therefore the corresponding part of its image formed by the telescope, and viewed by the eye subtends at the eye the $\angle \dfrac{\cdot 006}{13} . 20$ $= \cdot 00923$ in circular measure $= 31' 44''$ nearly.

23. A prism-shaped piece of glass whose transverse section throughout is a quadrant of a circle has the curved surface silvered. A narrow cylindrical beam of compound light is incident perpendicularly on one of its faces. Describe the successive appearances presented to an eye looking through the other face as the light is moved from the edge of the prism, in a plane perpendicular to it and passing through the eye, especially marking the order of the colours.

Shew that the limits of the distance of emergence from the edge of the prism for a ray of light whose refractive index into the prism is μ are

$$a\sqrt{\frac{\mu}{2\mu+2}} \text{ and } a\sqrt{\frac{\mu}{2\mu-2}}.$$

24. In Art. 68 prove that for rays of different refrangibility, the locus of the primary focus will be a curve of the third degree having a cusp at the point of incidence on the refracting surface.

25. In Newton's experiment (Art. 156), if the screen be placed in a plane perpendicular to the direction of light before incidence on the prism, prove that the length of the

spectrum, for a given position of the edge of the prism, will be proportional to

$$\frac{(\mu_v - \mu_r)\sin i}{\cos^2 D \cos (D + i - \phi) \cos \phi'},$$

i being the refracting angle, $\mu_v\, \mu_r$ the refractive indices for extreme rays, $\phi\, \phi'$ the angles of incidence and refraction at the first surface and D the deviation for the mean ray.

26. Shew that in spectrum observations the primary focus is to be taken, and that when the angles of incidence and emergence of green light (in Newton's experiment) are equal, v_1 is greater than u for red light, and less for violet.

On which side of the prism should a lens of the same material be placed to bring the foci of rays of different colours to the same distance from the prism?

27. A ray of light is refracted through a prism in a principal plane. Shew that if the dispersion of two neighbouring colours be a minimum

$$\frac{\sin (3\phi' - 2i)}{\sin \phi'} = 1 - \frac{2}{\mu^2}.$$

28. Shew that an eye looking through a glass wedge at a white field, will, if the ∠ of the wedge be within certain limits, have its view bounded by a red arch; and that the bounding rays, between emerging from the glass and reaching the eye, will lie on a cone of the fourth degree.

29. A ray of white light passes through a prism i in a principal plane; if the dispersion is a minimum, the angle of incidence is given by the equation

$$\sin^{-1}\left(\frac{2-\mu}{\mu^2}\sin\phi\right) + \tfrac{3}{2}\sin^{-1}\left(\frac{\sin\phi}{\mu}\right) = i.$$

30. A small pencil of three colours is directly incident from air on a glass spherical refractor. If μ, $\mu + \delta\mu$, $\mu + \delta'\mu$ denote the refractive indices for the colours from glass to air,

with similar notation for the distances of the geometrical foci from the surface, shew that

$$\delta\mu \,.\, \delta'\left(\frac{1}{v}\right) = \delta'\mu \,.\, \delta\left(\frac{1}{v}\right).$$

31. Prove that for minimum dispersion (of a pencil passing through a prism in a principal plane)

$$\mu^2 \sin(3\phi' - 2i) = (\mu^2 - 2)\sin\phi';$$

ϕ', i, μ having their usual meaning.

32. "Fraunhofer shewed that in a telescope with two lenses a very fine wire placed inside the instrument in the focus of the object-glass is seen distinctly through the eye-piece, when the telescope is illuminated with red light; but is invisible by violet light even when the eye-piece is in the same position." *Ganot's Physics.* Give the probable reason of this phenomenon. *It is explained by the achromatism of the eye being imperfect.* Art. 190.

33. A pencil of compound light passes excentrically through n thin lenses separated by finite intervals. If the system be achromatic then

$$\Sigma_0^n\left(\frac{\delta b_r}{b_r + f_r}\right) + \Sigma_0^n\left(\frac{\varpi_r b_r}{b_r + f_r}\right) = 0,$$

where b_r is the distance of the point of intersection of the axis of the pencil before incidence on the r^{th} lens (whose focal length is f_r and dispersive power ϖ_r) from the centre of that lens.

Obtain from this result the ordinary expression for the ratio of the dispersive powers of two lenses separated by an interval a for a pencil of light whose axis cuts the axis of the lenses at a distance b from the centre of the first lens, viz.

$$\frac{\varpi_1}{\varpi_2} = -\frac{(a+b)f_1 + ab}{bf_2 + ab}.$$

34. Having given $\mu_v = 1\cdot545$, $\mu_r = 1\cdot525$ and the focal length of a lens for rays of mean refrangibility $= 4$ *in.* and its breadth $= 2$ *in.*, shew that the diameter of the circle of chromatic aberration $= \cdot0128$ *in.*, nearly, for a pencil of rays parallel at incidence. (Art. 183.)

35. If a small pencil of light pass directly through a plate of thickness b, the index of refraction being $f\left(\frac{x}{c}\right)$, x being measured from the plane of incidence, and c varying slightly with the colour of the light,—shew that the chromatic aberration on emergence is

$$\left\{\frac{b}{c} \cdot \frac{1}{f\left(\frac{b}{c}\right)} - \int_0^b \frac{dx}{cf\left(\frac{x}{c}\right)}\right\} dc;$$

$f\left(\frac{0}{c}\right)$ being supposed equal to unity.

Chapter X.

Vision through Lenses, &c.

1. A person can see distinctly at the distance of 6 inches, find the focal length and nature of a lens which will enable him to see distinctly at a distance of 18 inches.

2. A concave lens is placed directly between an eye and a screen, determine how much of the screen will be visible through the lens.

3. A wafer is viewed through a convex lens, of 8 inches focal length, placed half-way between it and the eye; if the diameter of the lens be $\frac{1}{3}$ inch, that of the wafer $\frac{1}{2}$ inch, and the distance of the wafer from the eye 8 inches; the whole of the wafer will just be seen.

4. An eye is placed close to a sphere of glass, a portion of the surface of which, most remote from the eye, is silvered, —prove that assuming eight inches to be the least distance

of distinct vision, the eye cannot see a distinct image of itself unless the diameter of the sphere be at least ten inches in length.

5. A short-sighted person moves his eye-glass gradually from his eye towards a small object; shew that the linear magnitude of the image will keep increasing during the motion, and that the angle subtended by the image at the eye will be least when the eye-glass has advanced half-way towards the object.

6. At what distance from the eye must a concave lens be placed that the apparent linear magnitude of a small distant object may be diminished one-half?

7. Explain the following facts;—an object seen with both eyes appears single;—we form erroneous ideas of the size and distance of an object in an unusual situation;—it is very difficult to judge of the exact place of an isolated object seen with one eye only.

8. A small object is viewed through a sphere of water ($\mu = \frac{4}{3}$; radius of sphere $= \frac{1}{2}$ inch),—being placed at a distance of $\frac{1}{4}$ of an inch from the sphere; find the magnifying power.

9. A concave lens is moved from contact with a small object up to the eye, shew that the apparent magnitude of the image seen by the eye will first diminish and then increase,—and that it will be a minimum when it is midway between the object and the eye.

10. An object viewed through a convex lens in two different positions appears in each case equally magnified, but one image is erect and the other inverted,—shew that their mean distance from the lens is equal to its focal length.

11. An object is viewed through a convex lens in two different positions, so that in each case the image appears equally magnified, but in one case erect and in the other inverted; if the nearer position be determined by u, and f be the focal length, shew that the distance between the two positions is $= 2(f - u)$.

12. Shew that a defect in the eye of such a nature that the rays of a pencil incident in a horizontal plane are differently refracted from those in a vertical plane, may be remedied by the use of a lens of which one surface is cylindrical and the other spherical. Find the foci of a pencil of parallel rays in the two principal planes, when the radii of the sphere and cylinder are $3\frac{1}{2}$ and $4\frac{1}{2}$ inches respectively.

13. A closed hollow cylinder, about two inches long, has in the middle of one end a very small hole, and in that of the other a circular aperture of about the same diameter as the pupil of the eye. A pin is so placed in the plane of the aperture that its head is near the centre. When the flame of a candle is viewed through the tube, the aperture being held close to the eye, the flame appears upright and the pin inverted.

Explain the phenomenon.

Telescopes, &c.

14. The angular radius of the uniformly bright field of view in Galileo's telescope—with the usual notation—is

$$=\frac{f_e y_0 - f_0 y_e}{f_0 (f_0 - f_e)}.$$

15. The aperture of the object-glass of an astronomical telescope being 5 inches, and that of the pupil of the eye $\frac{1}{5}$ of an inch, find the least magnifying power for which the whole of a pencil incident on the object-glass can enter the eye.

16. The apertures of the eye-glass and object-glass of an astronomical telescope of which the magnifying power is 50, are respectively 3 and 2 inches; find the diameter of a stop that will completely intercept the ragged edge bordering the field of view.

17. What are the several effects of covering the central part of the object-glass, and of the eye-glass of an astronomical telescope?

18. Why is the apparent brightness of a star increased by the use of a telescope, whilst that of a planet is not?

19. In the simple astronomical telescope, when the apertures of the two lenses are proportional to their focal lengths, the field of view (as seen by whole pencils) is a single point.

If a convex lens of the same aperture as the eye-glass but of any given focal length, be placed in contact with the eye-glass, determine whether the field of view is thereby increased or diminished; (the telescope being always supposed adjusted for distant objects).

20. The focal length of the object-glass of a simple astronomical telescope is 15 inches, and its breadth is 3 inches; find the focal length of the eye-lens in order that on looking at a distant point the pencil of rays may just fill the pupil of the eye,—the breadth of the pupil being one-fifth of an inch.

21. In an astronomical telescope—with the usual notation—shew that for a person who can see distinctly at a distance a, the diameter of the stop should be

$$\frac{a(f_0 y_e - f_e y_0) + f_0 f_e y_e}{f_0 f_e + a(f_0 + f_e)}.$$

22. In an astronomical telescope, object-glass of 40 inches focal length, with an erecting eye-piece whose lenses beginning with the field-lens are 1·25, 2·1, 1, ·5 inches respectively;—the distances between the first and second lenses being 2 inches, and between the second and third $1\frac{1}{2}$ inches, and the distance between the field-lens and object-glass 41 inches,—find the magnifying power, and the position of the remaining lens when the instrument is in a perfect state of adjustment.

23. In Art. (200); the distance between the two positions of the object for which the images are magnified m times, one erect and the other inverted, is $\dfrac{2f}{m}$.

24. In Galileo's telescope, if F, $F_{\prime}$, $F_{\prime\prime}$, be the magnitudes of the field of view visible by half pencils, by whole pencils and of the entire field visible respectively, shew that

$$2F = F_{\prime} + F_{\prime\prime}.$$

25. What effect is produced by viewing an object very close to the eye, but through a fine pin-hole?

26. Three convex lenses of focal lengths $f_1 f_2 f_3$ are separated by intervals a, b, find the magnifying power of the combination, and prove that it will be independent of the position of the object if

$$(f_2 - a)(f_3 - b) + f_2 (f_1 + f_3 - a - b) = 0.$$

27. Explain why in looking through a moderately thick hedge, we obtain a better notion of the objects on the other side of the hedge by running alongside it, than by standing still.

28. A Galileo's telescope is adjusted so that a pencil from an object 289 feet from the object-glass emerges as a parallel pencil; the focal length of the object-glass is 1 *foot*, and that of the eye-glass 1 *inch*; shew if the axis is directed to the sun and a piece of paper held 23 inches from the eye-glass an image of the sun is formed on the paper. The sun's apparent diameter being $\cot^{-1} 120$, what is the size of this image, and is it erect or inverted?

29. A looking into B's eye sees four images of a candle. The first is very bright, small and erect—the second is fainter, larger and erect,—the third is still fainter smaller and inverted,—and the fourth is inverted, of a dull reddish brown colour, and indistinct, but rendered more distinct if A uses a concave glass. When B adjusts his eye to an object very close to his eye, the first image remains unaltered, the second and third are diminished, and the fourth requires a stronger concave glass to render it distinct. How are these images formed, and what do they indicate about the adjustment of the focal distance of the eye?

30. If the bright globe of a lamp be looked at for some time, and the eyes be then turned towards a dark part of the

room, the image remains, but changes colour from *yellow* through *orange*, *red* and *violet*, and disappears with a *greenish blue* tint. Explain this.

31. Shew how any telescope may be used for discovering approximately the distance (supposed not very great) of any visible objects. Graduate the micrometer screw by which the length of the axis of an astronomical telescope when used for such a purpose might be ascertained. Suggest means of obviating errors due to changes in form of the observer's eye, on the supposition that these changes are not instantaneous.

32. In an astronomical telescope with an eye-glass of focal length f, if the instrument be in adjustment for a person who sees distinctly at a distance a, shew that the distance through which it has to be moved for a person who sees distinctly at distance b is

$$\frac{(a-b)f^2}{(a+f)(b+f)}.$$

33. The focal lengths of the large and small mirrors of a Gregorian telescope being 36 and $1\frac{9}{10}$ inches, and the distance between them 38 inches, find the position of the image formed by the small mirror.

Also, if the focal length of the eye-glass be 1 inch, determine approximately how far the small mirror must be moved in order to adapt the telescope to an eye which sees most distinctly at a distance of 11 inches.

34. Calculate the magnifying power and field of view of a Gregorian telescope from the following data—focal lengths of large mirror, small mirror, field-glass, eye-glass = 24, 2, 3, 1 inches severally, distance of field-glass and eye-glass 2 inches, field bar aperture = $\frac{5}{8}$ inch.

35. The focal lengths of the large and small mirrors of a Cassegrain telescope are 24 and 1 inches, and the distance of their principal foci $\frac{1}{25}$ inch,—find the position of the eye-lens, the focal length of which is 1 inch, and the magnifying power.

36. The object-glass of an astronomical telescope has a focal length of 50 inches, and the focal length of each lens of the Ramsden's eye-piece is 2 inches; find the position of the eye-piece when adjusted for ordinary eyes, and the magnifying power of the telescope.

Result.—The distance between the object-glass and field-glass of the eye-piece $= 50 \cdot 5$ inches and the magnifying power $= \frac{100}{3}$.

37. The focal lengths of the larger and smaller mirrors of a Gregorian telescope are 32 and 3 inches, and the distance between their principal foci $= \frac{1}{4}$ inch; the focal lengths of the lenses of the Huyghens' eye-piece are 3 and 1 inch; find the relative position of the eye-piece and mirrors when the instrument is adjusted for ordinary eyes, and the magnifying power.

38. To an eye placed at the aperture of the large mirror in Gregory's telescope there will appear an inverted image of both mirrors near the smaller; and if the axis of the smaller be slightly disturbed the images will be shifted towards that part of it which is most inclined from the larger. Prove this property, and explain its use in the practical adjustment of the telescope.

39. In a Newtonian telescope the longer diameter of the small plane mirror is 2 inches, and the diameter of the object-mirror 8 inches, find approximately what fraction of each incident pencil is stopped.

40. The object-glass of an astronomical telescope has an aperture of 1 foot and the magnifying power of the instrument is 240. Shew that the brightness of the image is to that of the object as 1 : 25, if aperture of pupil be $\frac{1}{4}$ inch.

41. The diameters of the eye-glass and object-glass of an astronomical telescope are 1 inch and 6 inches, and their focal lengths 1 inch and 20 inches respectively. If the axis be pointed to a rod of indefinite length at a distance of 150 feet, how much of it will be seen through the telescope?

42. A Gregorian telescope being adjusted so that the pencils of rays emerge from the eye-glass in a state of paral-

lelism, shew that to suit an eye which sees distinctly at a distance a, the small mirror must be moved towards the larger one through a space $=\dfrac{x^2f'^2}{f^2(a+f')+xf'^2}$, where f, f' are the numerical focal lengths of the small mirror and eye-glass, and x the distance between the principal foci of the large and small mirrors.

43. In an astronomical telescope with Ramsden's eye-piece, F, f being the focal lengths of the object-glass and of each lens of the eye-piece,—the magnifying power of the instrument $=\dfrac{4F}{3f}$.

44. The eye-glass of a Gregorian has its focus at the centre of the face of the object-mirror: if the curvature of the small mirror be slightly changed, the necessary change of position of the small mirror is $=\dfrac{F}{F+2f}$ (*variation of focal length of small mirror*), nearly.

Also if the curvatures of both mirrors be slightly changed, find approximately the ratio between the variations of the focal lengths when no change of position is necessary.

Eye-pieces, Microscopes, &c.

45. The relative positions of the lenses of an achromatic erecting eye-piece of four lenses are given; one of the lenses having been lost indicate a method of calculating the power of the lens to be supplied.

46. The focal length of the eye-glasses in a Ramsden's and a Huyghens' eye-piece are as 2 : 1; shew that they are of equal *power*.

47. The focal length and aperture of the field-glass of a Huyghens' eye-piece are each $1\frac{1}{2}$ inches; determine the focal length and aperture of the object-glass of a telescope, with which this eye-piece would give a magnifying power of 200 —and a breadth for the emergent pencils of $\frac{1}{20}$ inch—and also determine the field of view.

48. Let O, A, B, C, D denote the lenses of an erecting astronomical telescope taken in order, O being the object-glass. Focal lengths 36, $1\frac{7}{8}$, $2\frac{1}{2}$, 2, $1\frac{1}{2}$ inches severally, $AO = 37{\cdot}5$, $AB = 2{\cdot}5$, $BC = 3$ inches severally; find the magnifying power when adapted for an eye whose distance of distinct vision is 8 inches. *Result*, $40\frac{5}{7}$.

49. A combination of two thin lenses in contact is used as a microscope—if it be achromatic, determine which of the lenses has the greater dispersive power.

50. A sphere of glass and another of water being placed in air, what must be the proportion of their radii, that their magnifying powers may be the same?

51. A Wollaston's doublet is formed of two lenses of focal lengths, f, $3f$ respectively, and is in adjustment for viewing a small flat uncovered object: shew that if a plate of glass whose thickness is $\frac{f}{10}$ be laid on the object the instrument may be readjusted, without altering the position of the lower lens, by increasing the distance between the lenses by $\frac{2(\mu-1)}{6\mu-1}f$.

52. A compound microscope is composed of two convex lenses of focal lengths 1 and 3 inches, separated by a distance of 2 inches, the *latter* lens being the eye-glass. A small object will be seen most distinctly if its distance from the field-lens $= \frac{1}{2}$ inch.

Compare also the angle which the image seen subtends at the eye with the angle which the object would subtend at the eye if placed at a distance of 8 inches from the eye.

53. If A be the breadth of *the pencil* falling on the object-glass or mirror of any optical instrument, e the breadth of the same pencil as it enters the eye, c the least distance from the eye at which the object can be seen distinctly without the instrument, and D the distance of the object from the object-glass or mirror,—shew that the magnifying power $= \frac{Ac}{eD}$.

Apply the foregoing expression to the compound microscope and the astronomical telescope,—and shew why it is necessary in high magnifying powers to have strong illumination of the object in the former instrument, and a large object-glass or mirror in the other.

54. If the angular distance between the sun's centre and a distant station be measured with a Hadley's sextant, and the index moved forwards through an angle equal to the angle between the axis of the telescope and a normal to the fixed mirror, without moving the sextant, the sun's light will be reflected from the moveable mirror to the distant station.

55. In the common astronomical telescope the true angular breadth of the field of view is 2α, and including the ragged edge it is 2β; shew that the illumination at a point of the ragged edge, whose angular distance from the axis is θ, is less than the illumination at any point of the true field of view in the ratio $\beta - \theta : \beta - \alpha$ nearly.

56. The stop of an astronomical telescope is correct when the telescope is in focus for objects at a distance u from the object-glass; shew that for any other distance u' there will be a ragged edge or a part of the true field cut off according as $u' >$ or $< u$: and that the angular breadth of this portion will be

$$\frac{(u' \sim u) ff'y}{\{u(f+f') - ff'\}\{u'(f+f') - ff'\}},$$

where ff' are the focal lengths of object-glass and eye-glass, and y the breadth of the eye-glass.

57. A person who reads small print at a distance of two feet finds that with a pair of plano-convex spectacles he can read it at a distance of one foot—what is the radius of the curved surface of either lens, $\mu = \frac{3}{2}$?

58. The lenses of a common astronomical telescope whose magnifying power is 16, and length from object-glass to eye-glass $8\frac{1}{2}$ inches, are arranged as a microscope to view an object placed $\frac{5}{8}$ of an inch from the object-glass; find the

magnifying power, the least distance of distinct vision being taken to be 8 inches.

59. Rays parallel to the axis fall on a reflecting curve $y = f(x)$; shew that the point of the caustic corresponding to x, y is

$$\xi = x + \frac{1}{2}\frac{\frac{dy}{dx}\left\{1 - \left(\frac{dy}{dx}\right)^2\right\}}{\frac{d^2y}{dx^2}}, \quad \eta = y + \frac{\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}.$$

If σ be the length of the caustic and $\sigma \pm x =$ constant, shew that the reflecting curve is $\frac{x}{c} = \log.\sin\frac{y}{c}$.

60. If the focal length of a Newtonian telescope be 2 *feet*, and the focal length of the eye-glass 1 *inch*, and if the instrument be in focus for a star to a person who sees most distinctly at a distance of 6 *feet*,—prove that it requires no readjustment for a person who sees most distinctly at a distance of 2 *feet* and is viewing an object whose distance is 600 *yards*.

61. In a Gregorian telescope if the focal length of the small mirror and of the eye-piece be each 2 *inches*, and the distance between the foci of the large mirror and the eye-piece be 32 *inches*—and the telescope be adjusted so that rays from a distant point emerge in a state of parallelism—then the alteration needed for a person who can see best at a distance of 26 *inches* will be a motion of the small mirror of approximately ·0005 inch.

62. In an astronomical telescope fitted with a Ramsden's eye-piece—shew that the radius of a stop which will intercept all but complete pencils will be the smaller of the two expressions $\frac{3}{4}r$ and $\frac{4Fr - fR}{4F + f}$, where F, R are respectively the focal length and radius of the object-glass, and f, r similar quantities for either of the two equal lenses which compose the eye-piece.

63. If there be two circular discs having slits made in them along the curve defined by the equation $r = f(\theta)$, the curves being turned in opposite directions, and if the discs be made to revolve in opposite directions about a common axis through their poles with angular velocities, ω, $n\omega$ respectively, shew that the appearance on looking at a bright sky through this apparatus is that of $n+1$ curves defined by the equation

$$r = f\left(\frac{n+1}{n-1}\theta\right).$$

Explain the result when $n = 1$.

64. Two circular discs, the planes of which are parallel, rotate in opposite directions with angular velocities ω, $n\omega$ about a common axis. In the former are pierced n slits in the direction of radii which meet the circumference at the angular points of a regular polygon; on the other is traced a curve of the form given by the equation $r = f\left(\frac{\theta}{n+1}\right)$. Shew that if $\frac{2\pi}{n\omega}$ be less than the time during which an impression remains on the retina, a person looking at the latter disc through the slits on the former will see $n+1$ similar and equal curves of the form represented by the equation $r = f(\theta)$.

This and the preceding problem explain the principle of the *Anorthoscope*.

Chapter XI.

Of the Rainbow.

1. Prove the following formulæ for the radius of the primary rainbow (δ), and of the secondary rainbow (ϵ):

$$\sin\frac{\delta}{2} = \frac{(4-\mu^2)^{\frac{3}{2}}}{3\mu^2\sqrt{3}}, \quad \sin\frac{\epsilon}{2} = \frac{\mu^4 + 18\mu^2 - 27}{8\mu^3},$$

μ being the refractive index for water.

2. If $\alpha =$ altitude of sun's centre, $\beta =$ radius of any colour of a rainbow, γ the angle subtended at the eye of the observer by the distance between the two points where the bow of this colour meets the horizon, then

$$\sin\frac{\gamma}{2} \cdot \cos\alpha = \sqrt{\{\sin(\beta-\alpha)\sin(\beta+\alpha)\}}.$$

3. If bubbles of air were rising in water, would a fish see a bow corresponding to a rainbow?

If drops of liquid of mean refractive index 2 were falling in the air what would be the order of colours in the bow which would be formed?

4. A small pencil of homogeneous light proceeding from a point Q is refracted through a sphere (centre C), with one internal reflexion—if the emergent rays are parallel in the primary plane, then

$$\frac{u}{a} = \cos\phi \frac{4\cos\phi - \mu\cos\phi'}{2\mu\cos\phi' - 4\cos\phi},$$

where ϕ ϕ' are the angles of incidence and refraction of the axis QP of the pencil which is incident at P, $QP = u$ and $CP = a$.

5. If bb' be the breadths of the p^{th} and q^{th} rainbows respectively and δ the sun's apparent diameter, shew that

$$b' - b = \left[\sqrt{\left\{\frac{(q+1)^2 - \mu^2}{(p+1)^2 - \mu^2}\right\}} - 1\right](b - \delta), \text{ nearly.}$$

6. The angular breadths of the primary and secondary rainbows at the eye of the observer are respectively,

$$\frac{1}{\mu}\left(\frac{4-\mu^2}{\mu^2-1}\right)^{\frac{1}{2}} . \delta\mu, \text{ and } \frac{2}{\mu}\left(\frac{9-\mu^2}{\mu^2-1}\right)^{\frac{1}{2}} . \delta\mu, \text{ nearly,}$$

where $\delta\mu$ is the difference of the refractive index for the extreme colours, and μ the index for mean rays.

7. When the rays emerge parallel after two refractions and one reflexion within the drop, shew that if ϕ denote the

angle of incidence, μ the refractive index and D the deviation, then

$$2\cos\phi = \mu\cos\left(\frac{\pi}{4} + \frac{\phi}{2} - \frac{D}{4}\right).$$

8. Having the following approximate data,

obliquity of ecliptic $= 23^\circ\ 30'$, latitude of London $= 51^\circ\ 30'$,

$$\text{for rain-water, } \cos^{-1}\sqrt{\left(\frac{\mu^2-1}{15}\right)} = 76^\circ\ 40',$$

$$\text{and } \cos^{-1}\frac{4}{\mu}\sqrt{\left(\frac{\mu^2-1}{15}\right)} = 46^\circ\ 40',$$

shew that in the latitude of London, no portion of a tertiary rainbow can be seen by an observer, whose back is turned towards the sun, if the sun be distant from the summer solstice by an angle greater than that determined from the equation

$$\sec\theta = 2\cos 11^\circ\ 30'.$$

9. Find between what hours of the day on the 21st March a rainbow will be visible to an observer at the equator —having given

$$\mu = \frac{4}{3}, \quad \log_{10} 2 = \cdot 3010300, \quad \log_{10} 3 = \cdot 4771213,$$

$$L\sin 21^\circ = 9\cdot 5546217.$$

10. How are *white* rainbows accounted for?

In consequence of the finite size of the sun's disc, the colours of the rainbows given by pencils of light from different points of it, overlap each other—and the traces of colour may under exceptional circumstances become very slight. For instance when the sun-light passing through a cloud of ice-crystals in the upper strata of the atmosphere, and being reflected at the surfaces of such crystals, reaches the eye after being refracted through rain-drops in a lower stratum of the atmosphere,—the source of light is spread over a large spherical angle—there is no sharp edge to the bow,—the bands are broader and fainter—there is little trace of colour—and the *rainbow* is *white* or nearly so.

Such phenomena are not of frequent occurrence. See an account of one seen by M. Cornu on November 28, 1883, and reported by him in the *Comptes Rendus*, Tome 97, p. 1530. No. 27 (31 December, 1883).

Miscellaneous Problems.

1. AO is a radius of a sphere (reflecting internally) whose centre is O, at the bisection of AO a luminous point is placed. Supposing light to fall first on the side of the sphere towards A, and calling v_{2n}, v_{2n+1} the distances from A of the $(2n)^{th}$, and $(2n+1)^{th}$ images, shew that

$$v_{2n} = \frac{4n-1}{4n} \cdot AO, \quad v_{2n+1} = \frac{4n+3}{4n+2} \cdot AO.$$

2. If a circular disc whose circumference is studded with bright points be made to roll with great rapidity within a circle of double the diameter, shew that the appearance will be presented of a number of rectilinear rays of light diverging from a common centre. Find the number of the rays.

3. A pencil of parallel rays is incident on the curved surface of a cylinder, and the reflected rays fall upon a screen which is parallel to the ends of the cylinder, shew that the area of the screen which is illuminated by reflexion is $\pi (a^2 - b^2) \tan^2 \alpha + 6c (a - b) \tan \alpha$,—where c is the radius of the cylinder, a, b the distances of its ends from the screen, and α the inclination of the incident pencil to the axis of the cylinder.

4. A man standing on the banks of the Cam, beside Trinity bridge, observes that the inverted image of the concavity of the arch receives his shadow exactly as a real inverted arch would do, if it were in the place where the image by reflexion appears to be. Explain this.

5. A ray is incident parallel to the axis of a polished prolate spheroid, and after two reflexions becomes again parallel to the axis. Shew that if $(x_1 y_1)$ $(x_2 y_2)$ be the two points of reflexion, $\frac{x_1 x_2}{a^4} = \frac{y_1 y_2}{b^4}$:—and find the points when the path of the ray in the spheroid is a rectangle.

6. A plane luminous ellipse throws light on a small plane area parallel to it and situated in the line drawn through

the centre of the ellipse perpendicular to it. If a, b are the semi-axes of the ellipse, and c the distance of the small area,

$$\text{the illumination} \propto \frac{ab}{\sqrt{\{(a^2+c^2)(b^2+c^2)\}}}.$$

7. A person having a very small fragment of a concave reflector, the principal radii of curvature of it being ρ, ρ', wishes to place it so that it may be considered part of a paraboloid of revolution, the positions of the focus and axis of which are given. Shew that it must be placed with its plane of least curvature passing through the focus, at the distance $\frac{1}{2}\sqrt{(\rho\rho')}$ therefrom, and with its tangent plane inclined to the axis of the paraboloid, and to its focal distance at an angle whose sine is $\sqrt{\left(\frac{\rho'}{\rho}\right)}$.

8. It has been remarked by writers on optics that "the concavity of the heavens appears to the eye to be a less portion of a spherical surface than a hemisphere,"—and that "the apparent distance of its parts at the horizon is generally between three and four times greater than the apparent distance of its parts overhead." How do you account for this appearance of the sky? By what method are the proportions here mentioned determined? Shew how these circumstances may be used in accounting for the apparent change in the size of the sun or moon, or in the distance of two neighbouring stars, as they ascend from the horizon to the meridian.

9. When a cylindrical china jar, standing upon the ground, receives the sun's rays obliquely, a bright curve is observed to form itself at the bottom of the jar, and it is found that the shape and dimensions of this curve are not affected by the varying elevation of the sun: account for this latter circumstance, and determine the nature of the bright curve.

10. A plane mirror, moveable about an axis in its own plane parallel to the axis of the earth, revolves from east to west with half the sun's apparent diurnal motion. Shew that the direction of the reflected rays of sunlight will not be sensibly altered during the day.

11. The sun's light is refracted through a prism, the edge of which is vertical,—find the position of the refracting surfaces in order that for a given altitude of the sun the deviation of the rays of a given colour may be a minimum.—

If z be the sun's zenith distance, i the refracting angle, x the angle of first incidence reduced to the horizon, μ the index for the given colour,—shew that the minimum deviation D is given by the equations

$$\sin \frac{D}{2} = \sin z \sin \Big(x -$$

$$\sin x = \mu \sin \frac{i}{2} \quad \sqrt{1 + \left(1 - \frac{1}{\mu^2}\right) \cot^2 z}$$

12. One half of a circle is a bright reflecting surface, the other half dull; a luminous point is placed at the bisection of the dull part. Shew that the bright surface is divided into two portions each of which is equally illuminated.

13. A narrow flat polished steel-wire is bent into the form of a circle, so as to form a cylinder of indefinitely small height; a luminous point is placed so that the perpendicular from it to the plane of the ring falls within the ring. Shew that all the rays after reflexion pass through a straight line of finite length. Calculate the length and position of this line, and point out the modification of the problem when the perpendicular does not fall within the ring.

14. If a plane surface be placed parallel to the plane of the ring—(Prob. 13)—and below it,—the bright curve on the plane surface has for its equation $r = a + b \cos \theta$, the point where the line passing through the luminous point and the centre of the ring meets the plane being taken as the pole.

Trace this curve and find the condition that it may have a cusp.

15. The directrix of a parabola is reflected at the curve, the exterior rim of which is polished; shew that the image is a curve lying between the parabola and its evolute and bisecting all the lines which are common normals to the one and tangents to the other.

16. A *small* plane area is illuminated by a bright circular ring of given dimensions,—the line joining the centres of the area and of the ring being perpendicular to the plane of the ring; find the position of the area, so that the illumination may be the greatest possible.—

If the area be replaced by a bright point, find its position so that the greatest quantity of light may be thrown on the ring.

17. A sphere of glass encloses a concentric sphere of an opaque substance and is placed on a horizontal table, a bright point is placed very near, but not close to its highest point; find the least magnitude of the opaque sphere which will prevent any ray from reaching the table;—if the magnitude be slightly less than this, find the diameter of the circular bright ring on the table.

18. A ray of light is reflected a number of times between two plane mirrors—not in the principal plane; prove that all the reflected segments of the ray are generating lines of a hyperboloid of revolution.

19. A plane mirror revolving about a vertical axis in its own plane, receives and reflects a small sunbeam, which after reflexion forms a spot of light on the horizontal floor. Determine the motion of the spot on the floor, and shew that its image as seen in the revolving mirror is stationary.

20. A small plane mirror is placed at the principal focus of a telescope, nearly perpendicular to its axis, and the telescope is directed approximately to a distant luminous object; shew that the rays reflected at the mirror will, after repassing the object-glass, return in the exact direction in which they came, in spite of the small errors of adjustment of the mirror and telescope.

21. A surface exposed to a luminous sphere is such that the illumination at any point $\propto$ (distance)$^{-n}$ of the point from the centre of the sphere. Examine the cases when $n=0, 1, 2, 3$.

22. A small pencil of rays emanates from a certain point on a parabola the axis of the pencil being an ordinate to the

curve. If the primary focus after reflexion at the curve be on the parabola, prove that the angle between the axis of the incident and reflected pencil is $2\sin^{-1}\frac{1}{\sqrt[4]{2}}$.

23. A ray proceeds from any point of the curve

$$\frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}=1,\ z=0$$

in any direction, and is reflected by the surface

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1;$$

shew that after reflexion its direction will again intersect the curve.

24. Find the direction in which light must be incident on one face of a triangular glass prism in a principal plane, in order that a ray after two internal reflexions and emergence at the same face, may proceed in the same direction as a ray directly reflected without entering the prism.

If the edge of the prism be placed parallel to the earth's axis, and the prism fixed so that the above direction is parallel to the plane of the meridian;—shew that when the sun's disc crosses the meridian, two images of the sun may be seen moving in opposite directions and crossing each other.

What is the principle and use of the *Dipleidoscope?*

25. An atmosphere consists of concentric spherical strata, the density in any stratum varying inversely as the square of its radius; prove that the path of a ray incident on a certain stratum at an angle whose secant is the refractive index of that stratum, will be a reciprocal spiral,—and that at *every* stratum the secant of the angle of incidence will be equal to the refractive index.

Also determine the path for a greater angle of incidence than the above.

N.B. If μ be the refractive index for a gas for a ray entering from a vacuum, it is found by experiment that $\mu^2-1 \propto$ density.

26. A small plane perpendicular to one of the diagonals of a cube at a point O in that diagonal produced is illuminated from the uniformly bright surface of the cube; shew that the illumination is measured by the expression

$$3I \cdot \frac{\theta}{\sin\theta} \sin\alpha \sin\beta \sin\frac{\pi}{3},$$

where θ is the angle between two adjacent edges of the pyramid (vertex O) circumscribing the cube, and α, β are their inclinations to the diagonal.

27. Two parallel and equal lines AB, CD are traced on paper, and viewed by the two eyes placed so that the line joining the centres of the eyes is parallel to AC; shew that the eyes may be so adjusted that the images of AB, CD may appear superposed, and that the resulting image will appear parallel to the plane of the paper, nearer to the eyes and smaller than AB or CD.

Under the same circumstances if AB, CD be inclined at a small angle, their images may appear superposed, and the resulting image will appear to *project* from the plane of the paper. Explain this, and shew why the parts of two pictures which, when viewed with the *stereoscope*, appear to project towards the eyes, frequently appear, when superposed by the naked eyes, to project in the opposite direction.

28. The shadow of a ball is cast by a candle upon an inclined plane in contact with the ball; prove that as the candle burns down, the locus of the centre of the shadow will be a straight line.

29. If the index of refraction of a medium at a point distant r from a fixed point be $\sqrt{1 \pm \frac{r^2}{a^2}}$, prove that a ray of light traversing the medium will describe a central conic section whose eccentricity will be > or < 1 according as the upper or lower sign be taken.

30. On what law of density will the total refraction of a ray of light through the earth's atmosphere be the same as through a homogeneous atmosphere?

31. A vertical window-bar of an opposite house, seen through a pane of glass, appears a zig-zag line; describe the inequalities in the thickness of the pane of glass indicated by the deviations from the vertical, and give reasons for your conclusion.

Rain is streaming down a pane of glass in parallel vertical lines; what will be the appearance of a circular arch seen through it?

32. The shadow of a given ellipsoid thrown by a luminous point on the plane which passes through two of the principal axes, has its centre on the curve in which the same plane intersects the ellipsoid; shew that the equation to the locus of the luminous point is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\frac{z^2}{c^2}-1\right.$$

where a, b the semiaxes of the ellipse through which the plane passes, and c the remaining axis, are taken for the axes of x, y, z respectively.

33. A polished hemisphere is placed with its base in contact with a plane, and a cylindrical pencil of rays falls upon its convex surface in a direction perpendicular to the plane: the illumination at any point of the plane is proportional to

$$\frac{\sin^3\phi}{2+\sin\phi},$$

where ϕ is the angle which the ray reflected to that point makes with the plane.

34. A uniformly bright cylindrical column, of radius a, and of indefinitely great height, stands upon a plane; shew that the illumination of a disc of the plane, of radius r, concentric with the column,

$$\propto \frac{\pi}{2}(r^2-a^2)+a\sqrt{(r^2-a^2)}-r^2\cos^{-1}\frac{a}{r}.$$

35. Denoting two mirrors by $\alpha=0$, $\beta=0$, what must be the angle between them, that the ray parallel to $\alpha-\beta=0$ may, after reflexion at each mirror, be parallel to $\alpha+\beta=0$?

36. The illumination at any point of an equilateral hyperbola, the centre being the origin of light, $\propto$ (distance)$^{-4}$.

37. Shew that the space-penetrating power in an astronomical telescope is measured by $\left(A \cdot \frac{f}{F}\right)^2$, where F and f are the focal lengths of the object and eye-glass respectively.

38. The magnifying power of a telescope is

$$= \frac{\text{breadth of visual pencil at the object-glass}}{\text{breadth of visual pencil at the eye-glass}},$$

supposing no light lost at any lens.

39. A blinking cat looks at a very little fish, which is in the axis of a thick closed glass cylinder full of water; first, when the axis of the cylinder is placed in a vertical position, and secondly when it is horizontal: in both cases the fish is at the same distance from the cat's eye, the line joining them being horizontal and perpendicular to the axis of the cylinder. Prove that the cat will believe the little fish to be less remote in the former than in the latter position of the cylinder:—and ascertain the difference between the apparent distances in terms of the thickness of the glass and the radius of the internal surface of the cylinder.

40. An erecting eye-piece of four lenses has the focal lengths of the lenses f_1, f_2, f_3, f_4 and their intervals a_1, a_2, a_3; if f is the focal length of the equivalent simple lens,

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3} + \frac{1}{f_4} + \frac{a_1}{f_1}\left(\frac{1}{f_2} + \frac{1}{f_3} + \frac{1}{f_4}\right) + a_2\left(\frac{1}{f_1} + \frac{1}{f_2}\right)\left(\frac{1}{f_3} + \frac{1}{f_4}\right)$$

$$+ \frac{a_3}{f_4}\left(\frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3}\right) + \frac{a_1 a_2}{f_1 f_2}\left(\frac{1}{f_3} + \frac{1}{f_4}\right) + \frac{a_1 a_3}{f_1 f_4}\left(\frac{1}{f_2} + \frac{1}{f_3}\right)$$

$$+ \frac{a_2 a_3}{f_3 f_4}\left(\frac{1}{f_1} + \frac{1}{f_2}\right) + \frac{a_1 a_2 a_3}{f_1 f_2 f_3 f_4}.$$

41. A luminous sphere rests within a hemisphere of twice its radius, the rim of which is horizontal; find the

whole illumination of the interior surface of the hemisphere;—and if the sphere be raised so that its lowest point just coincides with the centre of the hemisphere, shew that the illumination will be diminished in the ratio

$$\sqrt{5}-1 : \sqrt{5}+1.$$

42. A pencil of rays is refracted directly through a series of concentric spherical strata; if μ_0, μ_r be the indices of refraction from vacuum into air and into the r^{th} stratum (counting from the outermost), c_r the outer radius of the r^{th} stratum, and v_0, v_r the distances of the initial focus and of the focus after r refractions from the common centre, shew that

$$\frac{1}{\mu_r v_r} - \frac{1}{\mu_0 v_0} = {}^1\Sigma_r^r \left(\frac{\mu_{r-1} - \mu_r}{\mu_{r-1} \cdot \mu_r \cdot c_r}\right).$$

43. A ray of light is incident perpendicularly on one of the faces of a prism of which the density varies in such a manner that the coefficient of refraction at any point is $\mu\epsilon^\theta$,—μ being constant, and θ the angle which a plane through the point and the edge of the prism makes with that face upon which the ray is incident. If α be the refracting angle of the prism, ϕ the angle of incidence on the second face, shew that ϕ is determined by the equation

$$\cos\phi - \sin\phi = \epsilon^{\phi - 2\alpha}.$$

44. The focal lengths of the large and small mirrors of a Gregorian telescope are 18 and $\frac{19}{20}$ inches respectively, and their distance from each other is 19 inches; find the position of the image formed by the small mirror, and the magnifying power, if the focal length of the eye-glass $= \frac{1}{2}$ inch.

45. The density at any point of a prism (vertical $\angle = 2\beta$) varies as its distance from the nearest face of the prism. If a ray pass through it in a principal plane, its distance from the edge at the points of incidence and emergence being a, and its nearest approach to the edge being c, the deviation D is given by the equation

$$\sin\left(\beta + \frac{D}{2}\right) = \sin\beta \,.\, \epsilon^{\frac{Dc\sin\beta}{2(a - c\cos\beta)}}.$$

46. A spherical mirror is to be reduced by grinding to a parabolic one of the same focal length; find approximately what thickness parallel to the axis must be ground away at any point.

47. In Art. 122, if $\mu = f(x)$ shew that

(i) the curvature at any point of the trajectory

$$\propto \frac{d}{dx} \cdot \left\{\frac{1}{f(x)}\right\}.$$

(ii) Find the form of $f(x)$ in order that the trajectory may be a hyperbola.

(iii) If $f(x) = \alpha + \beta \sin \frac{x}{c}$, determine the trajectory.

48. A small pencil is *directly* refracted through a medium for which the index of refraction at any point is $\mu = \frac{ak}{d+x}$ (as in Art. 122). Shew that the distance of the geometrical focus from the origin of light will be

$$= \tau \left(1 - \frac{2d + \tau}{2ka}\right),$$

where τ is the whole thickness of the medium traversed.

49. The distance between a luminous point and the centre of the pupil of the eye is (D), and the radius of the pupil (r), which is supposed very small compared with D; shew that the illumination of the pupil is

$$\frac{c\pi r^2}{D^2} \cdot \cos \alpha \left(1 - \frac{3}{4} \frac{r^2}{D^2} + \frac{15}{8} \frac{r^2}{D^2} \sin^2 \alpha\right) \text{nearly},$$

where α is the angle which D makes with the axis of the eye, and higher powers of $\frac{r}{D}$ are neglected.

50. The axis of the object-glass of an astronomical telescope is inclined to the axis of the telescope at a small angle α; shew that in order to adjust the telescope for viewing distant objects, the eye-glass must be pushed in through a space

$$= \frac{\mu + 1}{2\mu} \cdot \alpha^2 \cdot f,$$

where f is the focal length of the object-glass, and μ its refractive index.

51. The extremity of the shadow of a vertical post standing on one side of a pond falls on a point A at the bottom of the pond. This point cannot be seen by a person on the opposite side, being just hidden by the sun's image in the water. Given the tan (*sun's altitude*) $= \frac{3}{4}$, the height of the post above the water $= 12$ feet, that of the eye $= 18$ feet, the breadth of the pond $= 70$ feet, and $\mu = \frac{4}{3}$; prove that the depth of the pond is 20 feet.

52. P, Q are the foci of incident and emergent rays of a small pencil passing directly through two thin coaxial lenses; shew that two fixed points M, N can be found in the axis of the lenses such that

$$\frac{1}{QN} - \frac{1}{PM} = \frac{1}{C};$$

and find C in terms of the focal lengths and distance of the lenses.

53. The ends of a glass cylinder are worked into portions of a convex and concave spherical surface,—radii r, s respectively,—having their centres in the axis of the cylinder; shew that the distance of these surfaces, in order that an eye placed at the concave surface may see the image of a distant object distinctly, must $= \frac{\mu(r-s)}{\mu - 1}$;—and the magnifying power will be $=$

54. If N, n, F be the focal centres and principal focus of a lens, the distances of the conjugate foci measured from two points N', n',—so situated in the axis that $nn' = p \,.\, NN' = (1-p) \,.\, nF$,—are connected by the equation

$$\frac{p}{v} - \frac{1}{pu} = \frac{1}{nF};$$

p being any constant quantity.

55. A hollow paraboloidal vessel, whose upper rim is circular, is placed with its axis vertical upon a horizontal plane, and exposed to the sun's rays. The boundary of the illuminated part of the interior of the vessel is a plane curve, whose projection on the plane of the rim is a circle of the same radius as the rim, and whose centre is distant from the centre of the rim a space equal to the latus rectum of the vessel multiplied by the tangent of the sun's altitude.

56. The distance between two lenses of a common astronomical telescope in perfect adjustment is a, the magnifying power being $\frac{q}{p}$. It is found that the object-glass can be achromatised by means of a certain lens placed in contact with it, and that if at the same time a concave lens of focal length b be placed in contact with the eye-glass the instrument will still be in adjustment; shew that if Δ_1, Δ_2 be the dispersive powers of the media forming the compound object-glass,

$$\frac{\Delta_1}{\Delta_2} = \frac{p^2 a}{(p+q)(pa-qb)}.$$

How is the character of the telescope altered by the above arrangement?

57. If a string be wrapped round a glass prism whose section is an equilateral triangle, so as to be always inclined at the same angle to the axis of the prism, the portions of the string seen by internal reflexion will appear to be parallel to the portions seen directly.

58. If the earth, supposed spherical, were covered to a depth h with water, h being small compared with r the earth's radius—shew that the height to which a person must be raised above the surface of the water in order to see as far below the horizon as when he was on the surface of the earth is $\frac{h^2}{2r(\mu^2-1)}$ nearly, μ being the index of refraction for water.

59. A ray of light emanating from an umbilicus of an ellipsoid is reflected at the surface to the opposite umbilicus: prove that the length of the path is constant, and equal to

twice the distance between the extremities of the greatest and least axes.

60. A trapezium $ABCD$, of which the side CD is parallel to BA, and the angles at A and B are each $60°$, is the base of a right prism of glass. Prove that the prism may be used as a stereoscope, if the observer look in at the face AB; one picture being in contact with the face BC and the other opposite to CD at a distance which is to the distance of C from AD as $1 : \mu$. Prove also that the magnitude of the picture will be the greatest possible when AB is four times CD.

61. A penny being placed with a flat side on an ordinary looking-glass and a candle lighted and placed on the same side of the looking-glass as the penny, but so that the perpendicular from it on the looking-glass does not meet the penny, part of the penny's principal image is observed to be much brighter than the rest by an eye so placed that the image of each point may be regarded as its geometrical focus. Explain this and determine the position of the shaded part of the image.

62. Q is a luminous point situated on the circumference of a perfectly reflecting circle, QP any incident ray, PQ' the chord in direction of the reflected ray, p the point of intersection of PQ' with the consecutive reflected ray; prove that $Q'p = 2Pp$.

63. In Art. 70—if the reflecting portion of the surface, the radius of which is r, be approximately circular of radius a, and if ϕ be the angle of incidence, u the distance of the point of incidence from the origin of light, shew that the radius of the circle of least confusion is equal to

$$\frac{au \sin^2 \phi}{u(1 + \cos^2 \phi) - r \cos \phi}.$$

64. An opaque sphere attached by a string to the bottom of a vessel of water floats just immersed, the surface being exposed to the rays of the sun, whose zenith distance is α.

Describe the form of the shadow at the bottom and the colours with which it is fringed.

If the water be just deep enough for the bottom to have no absolute shade, its depth below the sphere : radius of sphere :: $\cos \frac{1}{2} (\theta_1 + \theta_2) : \sin \frac{1}{2} (\theta_1 - \theta_2)$, $\theta_1 \theta_2$ being the apparent zenith distance of the sun to an eye under water for violet and red rays respectively.

65. If light of intrinsic brightness I pass through an absorbing medium bounded by planes perpendicular to its direction, the material of which is such that the absorption of a unit of brightness at a distance x from the plane on which the light is incident is $f(x)$ for a unit of thickness of the medium at the point—shew that the intrinsic brightness on emergence through the plate whose thickness is a is

$$Ie^{\int_0^a \log(1-fx)\,dx}.$$

66. A bright line is placed parallel to the axis of a polished cylinder. Shew that the curve which is seen on the cylinder by an eye placed in the plane xy at the point $\alpha\beta$, lies on the surface

$$b^2\{(\alpha+x)^2+(\beta-y)^2+z^2\}=(\beta x-\alpha y)^2\{(b-x)^2+y^2\}+b^2y^2z^2,$$

b being the distance between the bright line and the axis of the cylinder which is the axis of z.

67. There are two confocal reflecting ellipses; a ray proceeds from a point P of either of them in a direction passing through one of the foci, and is continually reflected between the curves. If after $2n-1$ reflexions it returns to the point P, the length of the path $= n$ times the difference of the major axes.

68. If CA the diameter and CP any chord of a lemniscate be reflecting mirrors, shew that a ray incident on CP at P in the direction of the tangent at P, will retrace its course after three reflexions.

69. A varnished sign-board swings under a vertical sun. Shew that the envelope of any particular reflected ray is a circle.

Also find the size of the bright patch on the ground for any position of the sign-board.

70. If the Moon were a transparent refracting globe, of focal length equal to the distance from the Earth, shew that during a total eclipse of the Sun, the light and heat at any place at the instant of totality would be increased about fourfold.

71. The north bank of a canal which runs east and west is vertical. The Sun is shining in the south and waves are travelling along the canal. Shew that on the bank will be seen a series of bright waves running along, whose shapes will be epicycloids, if the hollows of the canal waves be circular arcs.

What difference will be made by the altitude of the Sun?

72. Assuming that light of all colours has the same velocity in a vacuum, shew that by reason of the earth's atmosphere, refraction and aberration ought each to produce an infinitesimal spectrum in the image of a star.

Taking Cassini's hypothesis of a homogeneous atmosphere, find the position of a star for which these spectra destroy each other; shew that aberration = tan (zenith distance), and that if two colours unite so will all.

73. Four luminous points are placed in a medium in a plane perpendicular to its plane surface. In what positions will an eye placed close to the surface see only two images?

74. A ray of white light is incident in a principal plane on one side of a prism and emerges at the opposite side after reflexion at the base. Shew that if the ray of medium refrangibility suffer no deviation at the opposite side, the violet or red rays will emerge nearest to the edge, according as the side of the prism on which the ray falls is greater or less than the other.

75. In Art. 122—if the surfaces of equal density be such that in going from any one to the next in the plane of (xy)

the expression for the refractive index μ may be put into the form $\mu^2 = (x+a)f(y^2 e^{\frac{x}{a}})$, shew that the parabola $y^2 = 4ax$ is a possible path for a ray to travel in.

76. In front of a paraboloid of revolution which has its external surface polished, a circular ring of radius R is placed, with its plane perpendicular to the axis and its centre at the intersection of the axis and directrix of the generating parabola; shew that r being the radius of the image of the ring,

$$\left(\frac{R+2r}{3}\right)^3 = 4a^2(R-2r),$$

and that if R = latus rectum $= 4a$, then $r = a$.

77. If μ be the index of refraction from vacuum into air of the same density as that of the atmosphere at the distance r from the earth's centre, and if p be the value of

$$-\frac{d\mu}{\mu} : \frac{dr}{r},$$

at the earth's surface, D the difference between the refraction at the horizon and at a very small apparent altitude A, prove that

$$\frac{D}{A} = \frac{p}{1-p}.$$

78. The angles at the base of a triangular prism are $(\theta - \phi)$ and 2θ, where $\sin\theta = \mu\sin\phi$; a ray of light falls on the shorter side of the triangle; the angle of incidence is θ on the side of the normal next the vertex; shew that the ray after reflexion from the base and from the other side will emerge from the base in a direction parallel to its original direction, and that unless $\sin^2\theta > \sin\phi$, the second reflexion will not be total.

79. A convex lens of focal length a is placed at a distance a in front of a concave mirror of focal length b. Shew that an object and its image formed by rays which have passed twice through the lens and have undergone an intermediate reflexion, are always at equal distances on opposite sides of a point dis-

tant $a - \frac{u}{2b}$ in front of the lens, and that the image is equal to the object but inverted.

80. The solid angle subtended at an eye under water by a given portion of a very distant object is to the angle subtended by the same portion when the eye is out of water as $\cos\theta \sin\theta'$ to $\cos\theta' \sin\theta$, the angles of incidence and refraction being θ and θ'. Prove this, and also shew that the number of rays from each point of the object which enter the pupil of the eye under water, is to the number which enter it out of the water, supposing the aperture to be the same, as $\cos\theta'$ to $\cos\theta$.

Hence infer that the whole sky if uniformly bright will also appear uniformly bright to an eye under water: and that, while the apparent magnitude is diminished, the brightness, omitting loss of light by transmission, is increased, but in a less proportion than that of the diminution of magnitude.

81. In one of the faces of an isosceles prism of flint-glass a cavity is made bounded by a spherical surface, and in contact with it is placed a double convex lens of crown glass one of the surfaces of which exactly fits the cavity; a plano-convex lens of crown glass is placed in contact with the second face of the prism, the centres of the two lenses being in a plane perpendicular to the prism. A pencil is incident directly on one of the lenses and after internal reflexion from the back of the prism emerges directly from the other, find the condition of achromatism.

82. The sides of a quadrilateral which can be inscribed in a circle are reflective; if a luminous point be placed on a diameter of the circle bisecting at right angles one of the diagonals of the quadrilateral, prove that the distance between two of the images is equal to the distance between the other two.

Also prove that if the luminous point be placed at the intersection of the diagonals, a circle can be inscribed in the quadrilateral formed by the images. If d be the distance

between the centres of the two circles, R, r their radii, and θ the angle between the diagonals of the quadrilaterals, shew that

$$d^2 = R^2 - Rr \operatorname{cosec} \theta.$$

83. A small lens and a luminous object on its axis are moved in such a manner that a point of the image remains fixed in position, and it is found that the defect of the brightness at this point from the maximum varies as the square of the sine of the angle turned through by the line joining the lens and point. Shew that the lens describes a conic section. (See Art. 151.)

84. A brass plate grooved with an infinite number of concentric circles is placed horizontally in the sun; shew that an observer whose eye is in the vertical plane through the sun and the centres of the grooves will generally see a bright straight line and an arc of a circle on the plate. If the plate be not horizontal, what will be the appearance?

85. Two horizontal straight edges are held between a window and the eye, at a short distance from the eye. The upper one is fixed and is slightly further from the eye than the lower. If the lower be moved up parallel to itself when they approach each other, the upper appears to meet the lower; account for this.

86. In a Galilean telescope, if m be the magnifying power, f the focal length of the object-glass, $2a$, $2b$ the breadths of the object-glass and eye-glass, find the field of view as limited by half-pencils at least on the object-glass.

If this be 2α, and remain constant while the magnifying power receives a small increment δm, shew that the focal length of the object-glass must be *diminished* by the amount

$$\frac{a-b}{(m-1)^2} \cdot \cot \alpha \cdot \delta m.$$

87. A plane luminous curve in a medium (μ) is viewed by an eye placed at the plane bounding surface. The eye

being the origin and the initial line the normal to the surface, shew that, if $\rho = f(\sin\theta)$ be the polar equation of the curve,

$$\rho = \frac{1}{2\mu} f\left(\frac{\sin\theta}{\mu}\right) \cdot \frac{2\mu^2 - (1+\mu^2)\sin^2\theta}{\mu^2 - \sin^2\theta}$$

is the equation of the image.

88. The axis of a telescope bisects at right angles a straight horizontal scale, one yard long divided into inches, and passes through a vertical axis 35 inches from the scale, in the plane of a mirror which is capable of turning about it. Determine the angle between two positions of the mirror in one of which the division marked 4, while in the other that of the division marked 33, appears on the cross wires of the telescope.

89. Four convex lenses whose focal lengths are a, b, b, a are placed at intervals $a+b$, $2b\dfrac{a+b}{a-b}$, $a+b$ on the same axis, shew that the emergent ray is in the same straight line with the incident ray.

90. In making with an Astronomical Telescope an observation for which it is essential that the brightness of the image on the retina should be at least a hundredth part of that of the object, shew that the highest magnifying power which can be obtained is 1000, the diameter of the object-glass being 25 inches, and that of the pupil of the eye $\frac{1}{4}$ inch.

What is the highest magnifying power that can be used without any diminution of brightness?

91. A transparent hollow cylinder stands on a table, and on the inside of the cylinder is wrapped a narrow band of bright reflecting foil in the form of a helix which just makes one revolution. In the centre of the upper rim of the cylinder is placed a luminous point: find the equation of the curve of light on the table, and trace the curve;—prove that

the illumination by reflected light at a distance r from the axis of the cylinder varies as

$$\frac{2a \pm r}{r} \frac{1}{\{4h^2 + (2a \pm r)^2\}^{\frac{3}{2}}},$$

where $2h$ is the height and a the radius of the base of the cylinder.

92. Given a curve and the origin of a pencil of rays, to find (i) whether the caustic curve has asymptotes, and (ii) the position of such asymptotes if they exist.

If a parabola have contact of the second order with a curve at a given point, its focus being S, shew that the caustic curve formed by reflexion of a pencil of rays diverging from S will have an asymptote corresponding to this point.

93. If θ, ϕ be the angles of incidence and emergence of two parallel rays passing through a prism in a principal plane; d_1, d_2 the distances between those rays before incidence and after emergence, shew that

$$\frac{d_1}{d_2} = -\frac{\delta\phi}{\delta\theta},$$

where $\delta\theta$ is any small change of θ and $\delta\phi$ the corresponding change of ϕ.

Shew from this that the position of minimum deviation is that of most distinct vision through a thin prism.

94. Find the position and radius of the circle of least confusion when a small pencil is obliquely and centrically refracted through a thin lens.

A double convex lens has faces whose radii are each 8 *inches* and its refractive index is $\frac{\sqrt{3}+1}{\sqrt{2}}$; find its power when its axis is inclined at 30° to the line of sight. (See Art. 112, Cor. 4.)

95. A luminous point is placed in contact with the base of a hemisphere of glass—refractive index μ and radius a; a sheet of paper is held parallel to the base on the other side of the hemisphere. Shew that if the distance of the luminous point from the centre of the hemisphere be $> \frac{a}{\mu}$, there will be a dark band on the paper bounded by two hyperbolas.

96. The image of a right line is formed by centrical pencils through a thin lens. If ρ_1 ρ_2 ρ_3 be the curvatures of the images formed by the primary focus, the circle of least confusion, and the secondary focus respectively, shew that

$$2\rho_2 = \rho_1 + \rho_3.$$

97. In a sphere of glass there is a cavity the boundary of which is a sphere described on a radius of the former. A small object is placed at the point where the glass is indefinitely thin, and is viewed by an eye at the other extremity of the diameter passing through this point—find where the image is formed and determine its magnitude. If $\mu = \frac{3}{2}$ shew that the image is one-fifth as large again as the object.

98. A and B are fixed points, A being a luminous point and B the nearest point of a glass sphere with refractive index μ. C a point on BA produced is the image of A as seen by an eye on AB produced beyond the sphere. Shew that AC is least when the radius of the sphere is

$$\frac{3\mu - 2}{2 - \mu} AB.$$

99. Light is incident upon a refracting medium, the index of refraction at any point of which is a function of the distance from a fixed plane; find the differential equation of the path. (See Art. 120—2.)

If the axis of x be perpendicular to the plane and the index of refraction be $\mu_0 \tan \frac{x}{a}$, shew that the equation of the

path of the ray which is incident at the point $\left(\frac{\pi a}{4}, 0\right)$ at an angle $\frac{\pi}{4}$ is

$$2 \sin \frac{x}{a} = c\epsilon^{\frac{y}{a}\sqrt{3}} + \frac{1}{3c}\epsilon^{-\frac{y}{a}\sqrt{3}}, \quad \text{where } c = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{6}}.$$

100. If a pair of lenses on the same axis be achromatic for rays incident on the first parallel to the axis, then

$$\frac{d}{d\mu_1}\left(\frac{1}{f_1}\right) d\mu_1 + \frac{d}{d\mu_2}\left(\frac{1}{f_2} - \frac{a}{f_1 f_2}\right) d\mu_2 = 0,$$

μ_1, μ_2, f_1, f_2 being the indices and focal lengths of the lenses, and a the distance between them.

101. If $T = 0, N = 0$ be the equations to the tangent and normal at any point P of a reflecting curve, ξ, η the co-ordinates of a radiant point, shew that the equation to the ray reflected from the curve at P is

$$\frac{T}{T'} + \frac{N}{N'} = 0,$$

where T', N' are what T and N become when ξ, η are substituted for the current co-ordinates.

102. Prove that the magnifying power of a thin double convex lens, the radius of each surface being ρ, when the space between the lens and an object at distance a is filled with fluid of index μ' is given by

$$\frac{1}{m} = 1 - \frac{a}{\rho} \cdot \frac{2\mu - \mu' - 1}{\mu'}.$$

103. Light parallel to the axis falls upon a convexo-plane lens; shew that the aberration for the extreme ray for refraction through the lens is equal to

$$\frac{(\mu - 1)^3 (\mu + 1) + 1}{2\mu^2 (\mu - 1)} \cdot \frac{y^2}{r},$$

where y is the radius of the circular plane face, and r that of the spherical surface.

104. A luminous point is placed on the axis of a concave lens at a distance u from it. The light falls on a screen at a distance κ behind the lens and perpendicular to the axis of the lens. If I is the illumination of the screen where it cuts the axis, and if I' is what the illumination would be if the lens were removed, shew that

$$\frac{I}{I'} = \frac{f^2 (u+\kappa)^2}{(fu + u\kappa + \kappa f)^2}.$$

105. A small pencil of parallel rays is refracted centrically through a double convex lens the radii of whose surfaces are each $= r$, and whose thickness is t; shew that, if the square of t be neglected, the distance of the primary focus from the point of emergence of the pencil will be

$$\frac{r \sin\phi' \cos^2\phi}{2 \sin(\phi - \phi')} - \frac{t \sin\phi' \cos^2\phi}{4 \sin\phi \cos^2\phi'},$$

ϕ, ϕ' being the angles of incidence and refraction.

106. Two conical shells of light given by the equation $z^2 = \pm xy$ are incident on the reflecting surface $xyz = const.$; shew that the reflected rays pass through two straight lines at right angles to each other.

107. Three lenses are placed so as to have a common axis, the second being equidistant from the first and third; if the combination be such that the image of a luminous point is always at the same distance from that point, prove that either the focal lengths of the first and third are equal, or the focal length of the second is equal to a quarter of the distance between the first and third. In each case find the constant distance, and in the second case shew that it is equal to the distance between the first and third lenses.

108. Rays are incident from the centre upon a reflecting ellipsoid at points situated on a central circular section: prove that the reflected rays pass through a diameter of the ellipsoid perpendicular to this section.

109. A portion of a medium, the refracting index of which at any point whose distance from a fixed point O is r is $\frac{c^2}{r^2}$, is bounded by three planes mutually at right angles passing through O: a ray of light is incident on one of the plane faces at a given point, find the length of its path in the medium.

110. A pencil travelling in a medium is refracted through n other media (whose boundaries are concentric spheres), if p, p' be the distances from the centre of the incident and emergent pencils, then

$$\frac{1}{2}\left(\frac{1}{p}-\frac{1}{p'}\right)=\frac{1}{r_1}+\frac{r_1-r_2}{\mu_1 r_1 r_2}+\frac{r_2-r_3}{\mu_2 r_2 r_3}+\ldots+\frac{r_{n-1}-r_n}{\mu_{n-1} r_{n-1} r_n}-\frac{1}{\mu_n r_n},$$

where $\mu_1, \mu_2, \ldots \mu_n$ are the refractive indices from the original medium into the others commencing from the outside, $r_1, r_2, \ldots r_n$ the radii of the bounding surfaces.

111. Shew that of a pencil from a point in the ragged edge of the field of view of an astronomical telescope there is lost a portion whose breadth is equal nearly to $a \cdot \psi$, where a is the distance between the lenses, and ψ is the angle subtended at the centre of the object-glass by the distance of the point from the boundary of the field of view.

112. A plane mirror and a concave mirror are placed opposite one another on the same axis at a distance apart greater than the radius of the concave mirror: a person standing with his back to the plane mirror, but close to it, observes the three brightest images of a candle he holds in his hand: he moves the candle forward, till it coincides with the nearest image, prove that the other two images will coincide also at the same time. He then moves the candle still further forward a distance x, till it coincides with another image; prove that at this instant the first image will disappear, and if a be the distance between the mirrors the radius of the concave mirror is

$$\frac{x+a \pm \sqrt{(x+a)^2-8ax}}{2}.$$

113. The focal length of the object-glass of an astronomical telescope is 40 inches, and the focal length of four convex lenses forming an erecting eye-piece are respectively $\frac{3}{2}$, $\frac{1}{6}$, $\frac{3}{8}$, $\frac{3}{2}$ inches, the intervals between the first and second, and between the second and third being 1 inch and $\frac{1}{2}$ inch respectively; find the position of the eye-lens, and the magnifying power when the instrument is in adjustment.

114. A man standing on the seashore sees the light of a star reflected on the surface of the sea, when it is covered with gentle ripples travelling in all directions; find the equation to the boundary of the bright patch on the water, considering the undisturbed surface of the sea to be a horizontal plane.

Find the condition that this patch should reach to infinity.

If z be the zenith distance of the star and tangents to the patch from the man's feet contain an angle 4θ, shew that if the patch does not extend to infinity, the angle which it subtends at the man's eye in the vertical plane passing through the star is $4 \tan^{-1} (\sin \theta \tan z)$.

115. A luminous point Q moves round a closed curve surrounding the focus H of an ellipse, and is seen by reflexion at the ellipse by an eye at the other focus S. Prove that two images will always be seen, and that if q be an image seen by reflexion at the ellipse at P,

$$\frac{SP \cdot Pq}{Sq} = \frac{HP \cdot PQ}{HQ}.$$

Also, if H be the centre of the curve traced out by Q, and a line through S meet the loci of the two images in q_1, q_2, and SQ' be taken on this line a harmonic mean between Sq_1, Sq_2, Q' will trace out a fixed curve whose area is to that of the ellipse as $2 + 3e^2 : 8$, where e is the eccentricity of the ellipse.

116. A pencil of rays is incident directly on the plane surface of a medium, the other boundary of which is such

that the length of the path of every ray within the medium is c: prove that the distance of the geometrical focus of the pencil after emergence from the origin of the pencil is

$$\frac{ac(\mu-1)^2}{\mu^2 a + c};$$

where μ is the refractive index, and a the distance of the origin from the first surface of the medium.

117. A ray of light passes through a medium whose index of refraction varies continuously; prove that

$$\frac{d}{ds}\left(\mu\frac{dx}{ds}\right) = \frac{d\mu}{dx},$$

s being the length of the path of the ray to a point whose co-ordinates are xyz.

If in air $\mu - 1$ varies as the density and if μ at a certain place is $\frac{3400}{3399}$, and if the height of the homogeneous atmosphere be five miles, prove that when the temperature is constant the effect of refraction on distant horizontal objects is to increase the earth's apparent radius as found from the dip from 4000 to 5230 miles: and that if the temperature over a frozen sea increase about 6^0 F. for every hundred feet of ascent, objects may be seen reflected in the sky.

118. The equation of the boundary of a comet or nebulous body being given, together with the density at any point of its interior, it is required to find the law of brightness of its apparent disc, supposing that each particle sends the same quantity of light to the eye of a distant spectator.

Ex. A spherical body of radius a, the density at any distance r from the centre being $\Delta\sqrt{(a^2 - r^2)}$.

Conversely. From the law of brightness of the apparent disc, and the density at any point and known distance of the body, find the equation of the boundary,—supposing it to be symmetrical with reference to a plane perpendicular to the direction of vision.

Ex. The density of the body being the same as in the foregoing example, the brightness at any apparent distance R from the centre of the disc varies as $a^2 - m^2R^2$.

See *Quart. Jour. Math.* Vol. III. p. 364.

119. In the phenomenon of Solar Halos how are the positions and colours of the inner and outer circles surrounding the Sun, and the positions of the Parhelia, theoretically explained? Why are the Parhelia not exactly coincident with the inner circle?

120. Within a reflecting circle on the same side of the centre are two parallel rays, one dividing the circumference into arcs which are as 3 : 1,—the other dividing it into arcs which are as 8 : 1; find the least value of n such that after each ray has suffered n reflexions, they may be again parallel.

121. Given the directions of three plane mirrors in space, construct a straight line such that if light from it be reflected by the three mirrors in succession, the third image shall be parallel to the straight line.

Walton and Mackenzie, *Solutions of the Cambridge Problems.* 1854, p. 73.

122. Rays are incident parallel to the axis of x on a reflecting curve, and the equation to the catacaustic is $y = f(x)$, the equation to the reflecting curve will result from eliminating t from the two equations

$$x - t = \frac{y - f(t)}{f'(t)} = \frac{C - \int [1 + \sqrt{\{1 + (f'(t))^2\}}]\, dt}{1 + \sqrt{[1 + \{f'(t)\}^2]}}.$$

Boole's *Differential Equations*, p. 257.

123. Supposing a very large number of hexagonal crystals of ice to descend continually in the atmosphere with their axes vertical and faces turned in all possible directions,—prove that the reflexions of the light of the sun or moon from the vertical faces will cause the spectator on the earth's surface to see a horizontal circle of white light of the same breadth and altitude as the luminary. What phenomenon may be explained by this result?

124. When a large number of crystals of ice of the form of hexagonal prisms with plane ends perpendicular to their axes are suspended in the air, it is found that many of the simple crystals are joined together at their ends or sides so that the axes are parallel. Hence shew that (i) those of the compound crystals which have the axes horizontal, and are situated opposite to the sun at the same altitude, send light to a spectator by two reflexions at vertical planes inclined at an angle of 90° to each other, whatever be the orientation of the planes; (ii) those which have the axes vertical, and are situated in an azimuth of 120° from the sun at the same altitude, send light to the spectator by reflexions at planes inclined to each other at an angle of 120°.

What phenomena observed sometimes to accompany Solar Halos are explained by these results?

CAMBRIDGE: PRINTED BY C. J. CLAY, M.A. & SON AT THE UNIVERSITY PRESS.